SYMMETRIC AND ANTISYMMETRIC VECTOR-VALUED JACK POLYNOMIALS

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Abstract. Polynomials with values in an irreducible module of the symmetric group can be given the structure of a module for the rational Cherednik algebra, called a standard module. This algebra has one free parameter and is generated by differential-difference ("Dunkl") operators, multiplication by coordinate functions and the group algebra. By specializing Griffeth's (arXiv:0707.0251) results for the G(r,p,n) setting, one obtains norm formulae for symmetric and antisymmetric polynomials in the standard module. Such polynomials of minimum degree have norms which involve hook-lengths and generalize the norm of the alternating polynomial.

1. Introduction

Hook-lengths of nodes in Young tableaux appear in a variety of different settings. Griffeth [5] introduced Jack polynomials whose values lie in irreducible modules of the complex reflection group family $G(r,p,N)$. This class of polynomials forms an orthogonal basis for the associated standard module of the rational Cherednik algebra. In this paper we specialize his results to the symmetric group and show how the norms of two special symmetric and antisymmetric polynomials in the standard module depend on the hook-lengths of the partition associated to the representation.

For $N \geq 2, x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and let $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$. For $a, b \in \mathbb{N}_0$ and $a \leq b$ let $[a, b] = \{a, a+1, \ldots, b\}$ (an interval of integers). The cardinality of a set E is denoted by $\#E$. For $\alpha \in \mathbb{N}_0^N$ (a *composition*) let $|\alpha| := \sum_{i=1}^N \alpha_i$, $x^\alpha := \prod_{i=1}^N x_i^{\alpha_i}$, a monomial of degree $|\alpha|$. The spaces of polynomials, respectively homogeneous, polynomials are

$$\mathcal{P} := \operatorname{span}_{\mathbb{F}} \left\{ x^\alpha : \alpha \in \mathbb{N}_0^N \right\},$$
$$\mathcal{P}_n := \operatorname{span}_{\mathbb{F}} \left\{ x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = n \right\},\ n \in \mathbb{N}_0,$$

where $\mathbb{F}$ is a field $\supset \mathbb{Q}$. Consider the symmetric group $\mathcal{S}_N$ as the group of permutations of $[1, N]$. The group acts on polynomials by linear extension of $(xw)_i = x_{w(i)}, w \in \mathcal{S}_N, 1 \leq i \leq N$, that is, $wf(x) := f(xw), f \in \mathcal{P}$. For $\alpha \in \mathbb{N}_0^N$ let $(w\alpha)_i = \alpha_{w^{-1}(i)}$, then $w(x^\alpha) = x^{w\alpha}$. Also $\mathcal{S}_N$ is a finite reflection group whose reflections are the transpositions (i, j); $x(i, j) = \left(\ldots, \overset{i}{x_j}, \ldots, \overset{j}{x_i}, \ldots \right)$. The simple reflections $s_i := (i, i+1), 1 \leq i < N$, generate $\mathcal{S}_N$.

Say $\lambda \in \mathbb{N}_0^N$ is a partition if $\lambda_i \geq \lambda_{i+1}$ for all i. Denote the set of partitions by $\mathbb{N}_0^{N,+}$. Suppose τ is a partition of N, that is, $|\tau| = N$; then there is an associated

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Ferrers diagram, namely the set of lattice points $\left\{(i,j)\in\mathbb{N}_0^2 : 1\leq i\leq\ell(\tau), 1\leq j\leq\tau_i\right\}$, also denoted by τ; the *length* of τ is $\ell(\tau) := \max\{i : \tau_i > 0\}$. The conjugate partition τ' is the partition whose diagram is the transpose of the diagram of τ (that is, $\tau'_m = \#\{i : \tau_i \geq m\}$. For a node (or point) $(i,j)\in\tau$ the *arm-length* is $\operatorname{arm}(i,j) := \tau_i - j$, the *leg-length* is $\operatorname{leg}(i,j) := \tau'_j - i$, and the *hook-length* is $h(i,j) := \operatorname{arm}(i,j) + \operatorname{leg}(i,j) + 1$. We will use $\operatorname{arm}(i,j;\tau)$ etc. if it is necessary to specify the partition.

To each partition τ of N there is an associated irreducible $\mathcal{S}_N$-module V_τ. We analyze the space $M(\tau)$ of V_τ-valued polynomials under the action of differential-difference ("Dunkl") operators. There is a canonical symmetric bilinear (the contravariant) form $\langle\cdot,\cdot\rangle$ on this space. We will construct distinguished polynomials $f^s_\tau, f^a_\tau \in M(\tau)$, with f^s_τ being symmetric and f^a_τ being antisymmetric, such that

$$\langle f^s_\tau, f^s_\tau\rangle = c_0 \prod_{(i,j)\in\tau} (1 - h(i,j)\,\kappa)_{\operatorname{leg}(i,j)},$$
$$\langle f^a_\tau, f^a_\tau\rangle = c_1 \prod_{(i,j)\in\tau} (1 + h(i,j)\,\kappa)_{\operatorname{arm}(i,j)};$$

and $c_0, c_1 \in \mathbb{Q}$ are constants depending on τ (the *Pochhammer symbol* is $(t)_n := \prod_{i=1}^n (t+i-1), n\in\mathbb{N}_0$). This result generalizes the situation of the trivial representation of $\mathcal{S}_N$; in this case $\tau = (N), f^s_\tau = 1, f^a_\tau = \prod\limits_{1\leq i<j\leq N}(x_i - x_j)$ and $\langle f^a_\tau, f^a_\tau\rangle = c_1\prod_{i=2}^N (1+i\kappa)_{i-1}$.

Section 2 collects the needed information about representations of $\mathcal{S}_N$. The Dunkl operators and their action on monomials are discussed in Section 3. The (nonsymmetric) Jack polynomials are constructed in Section 4; this material is the specialization of Griffeth's results for $G(r,p,N)$ to $\mathcal{S}_N = G(1,1,N)$. Our main results on symmetric and antisymmetric polynomials are contained in Section 5. There is the description of an orthogonal basis and the detailed exposition of the special polynomials whose norms involve the hook-lengths. In fact these are the symmetric and antisymmetric polynomials of minimum degree.

2. Representations of $\mathcal{S}_N$

Let $Y(\tau)$ be the set of *reversed standard Young tableaux* (RSYT) of shape τ, namely, an assignment of the numbers $\{1,2,\ldots,N\}$ to each node of τ such that entries decrease in each row and in each column. The node of T containing i is denoted $T(i)$ and the row and column of this node are denoted by $\operatorname{rw}(i,T), \operatorname{cm}(i,T)$ respectively, $i\in[1,N]$. The *content* of $T(i)$ is $c(i,T) := \operatorname{cm}(i,T) - \operatorname{rw}(i,T)$. Thus $c(N,T) = 0$ for each T. The well-known hook-length formula asserts that $\#Y(\tau) = N!/\prod\limits_{(i,j)\in\tau} h(i,j)$. Following Murphy [7] define an action of $\mathcal{S}_N$ on the $\#(Y(\tau))$-dimensional vector space $V_\tau := \operatorname{span}_{\mathbb{F}}\{v_T : T\in Y(\tau)\}$ as follows:

Proposition 1. *Suppose $T\in Y(\tau)$ and $b_i(T) := 1/(c(i,T) - c(i+1,T))$ for $1\leq i<N$ then:*
1) if $b_i(T) = 1$ (when $\operatorname{rw}(i,T) = \operatorname{rw}(i+1,T)$) then $s_i v_T = v_T$;
2) if $b_i(T) = -1$ (when $\operatorname{cm}(i,T) = \operatorname{cm}(i+1,T)$) then $s_i v_T = -v_T$;

3) if $0 < b_i(T) \leq \frac{1}{2}$ *(when* $\mathrm{rw}(i,T) < \mathrm{rw}(i+1,T)$ *and* $\mathrm{cm}(i,T) > \mathrm{cm}(i+1,T)$*then*

$$s_i v_T = b_i(T) v_T + v_{s_i T},$$
$$s_i v_{s_i T} = \left(1 - b_i(T)^2\right) v_T - b_i(T) v_{s_i T};$$

4) if $-\frac{1}{2} \leq b_i(T) < 0$ *(when* $\mathrm{rw}(i,T) > \mathrm{rw}(i+1,T)$ *and* $\mathrm{cm}(i,T) < \mathrm{cm}(i+1,T)$*) then* $s_i v_T = b_i(T) v_T + \left(1 - b_i(T)^2\right) v_{s_i T}$*, and* $s_i v_{s_i T} = v_T - b_i(T) v_{s_i T}$*.*
In cases (3) and (4) the tableau $s_i T$ *is obtained by interchanging the entries* $i, i+1$*. Furthermore in case (3)* $s_i f_0 = f_0$ *and* $s_i f_1 = -f_1$ *for*

$$f_0 = (b_i(T) + 1) v_T + v_{s_i T},$$
$$f_1 = (b_i(T) - 1) v_T + v_{s_i T}.$$

There is an ordering on tableaux such that $T > s_i T$ in case (4).

Corollary 1. *Let* $f = \sum_{T \in Y(\tau)} k_T v_T$ *with the coefficients* $k_T \in \mathbb{Q}$ *and* $s_i f = \pm f$ *for some* $i \in [1, N-1]$*. Then*
1) $T, s_i T \in Y(\tau)$ *implies* $k_{s_i T} = r k_T$ *for some* $r \neq 0$;
2) $s_i f = f$ *and* $\mathrm{cm}(i,T) = \mathrm{cm}(i+1,T)$ *implies* $k_T = 0$*;*
3) $s_i f = -f$ *and* $\mathrm{rw}(i,T) = \mathrm{rw}(i+1,T)$ *implies* $k_T = 0$.

Statement (1) means that $k_{s_i T}$ and k_T are either both nonzero or both zero.

Definition 1. *The Jucys-Murphy elements (in the group algebra* $\mathbb{Q}\mathcal{S}_N$*) are*

$$\omega_i := \sum_{j=i+1}^{N} (i,j), 1 \leq i \leq N.$$

There are commutation relations: $\omega_i \omega_j = \omega_i \omega_j$ for all i, j; $\omega_i s_j = s_j \omega_i$ for $j \neq i-1, i$; $s_i \omega_i - \omega_{i+1} s_i = 1$ (see Vershik and Okounkov [8, Section 4] for the representations of the algebra generated by $\{\omega_i, \omega_{i+1}, s_i\}$). Murphy proved the following:

Theorem 1. *Suppose* $T \in Y(\tau)$ *and* $i \in [1, N]$ *then* $\omega_i v_T = c(i,T) v_T$*.*

Let $\langle \cdot, \cdot \rangle_0$ be a $\mathcal{S}_N$-invariant positive-definite linear form on V_τ, (the form is unique up to a multiplicative constant) then each ω_i is self-adjoint and hence the vectors v_T are pairwise orthogonal, being eigenvectors with different eigenvalues. Denote $\|v\|_0^2 = \langle v, v \rangle$ For given T and i as in case (4) we have $\|v_T\|_0^2 = b(i,T)^2 \|v_T\|_0^2 + \|v_{s_i T}\|_0^2$ (since s_i is an isometry) and thus $\|v_{s_i T}\|_0^2 = \left(1 - b(i,T)^2\right) \|v_T\|_0^2$. There is one formula for $\|v_T\|_0^2$ in [7, Thm. 4.1]. The following is based on the content vector of T (that is, $(c(1,T), \ldots, c(N,T))$) :

Definition 2. *For* $T \in Y(\tau)$ *let*

$$\|v_T\|_c^2 = \prod_{1 \leq i < j \leq N,\ c(i,T) \leq c(j,T) - 2} \frac{(c(i,T) - c(j,T))^2 - 1}{(c(i,T) - c(j,T))^2}.$$

Lemma 1. *Suppose* $\{g_{ij}(T) : 1 \leq i < j \leq N\}$ *is a collection of functions on* $Y(\tau)$ *and satisfy (1)* $g_{ij}(T) = g_{ij}(s_m T)$ *for all* i, j *with* $\{i, j\} \cap \{m, m+1\} = \emptyset$*, (2)* $g_{i,m}(T) = g_{i,m+1}(s_m T)$ *and* $g_{i,m+1}(T) = g_{i,m}(s_m T)$ *for* $i < m$*, (3)* $g_{m,j}(T) =$

$g_{m+1,j}(s_mT)$ and $g_{m+1,j}(T) = g_{m,j}(s_mT)$ for $j > m+1$ for all $T \in Y(\tau)$ and $m \in [1, N-1]$ such that $s_mT \in Y(\tau)$, then

$$\frac{\prod_{1\le i<j\le N} g_{ij}(T)}{\prod_{1\le i<j\le N} g_{ij}(s_mT)} = \frac{g_{m,m+1}(T)}{g_{m,m+1}(s_mT)}.$$

The proof is a straightforward calculation.

Proposition 2. *Suppose $0 < b_i(T) \le \frac{1}{2}$ for $T \in Y(\tau)$ and some $i \in [1, N-1]$ then $\|v_{s_iT}\|_c^2 = \left(1 - b_i(T)^2\right)\|v_T\|_c^2$. Thus $\|\cdot\|_c$ is an $\mathcal{S}_N$-invariant norm.*

Proof. By hypothesis $c(i,T) \ge c(i+1,T)+2$ and $c(i, s_iT) = c(i+1,T)$, $c(i+1, s_iT) = c(i,T)$. In the ratio $\|v_{s_iT}\|_c^2 / \|v_T\|_c^2$ all factors except $\left(1 - \left(\frac{1}{c(i+1,T)-c(i,T)}\right)^2\right)$ in the numerator cancel out, by Lemma 1. □

Henceforth we drop the subscript "c" and use "0" for the form. Next we consider invariance properties for certain subgroups of $\mathcal{S}_N$. Specifically these are the stabilizer subgroups of a monomial x^λ, where $\lambda \in \mathbb{N}_0^{N,+}$.

Definition 3. *For $1 \le a < b \le N$ let $\mathcal{S}_{[a,b]} = \{w \in \mathcal{S}_N : i \notin [a,b] \Longrightarrow w(i) = i\}$, the subgroup of permutations of $[a,b]$, generated by$\{s_i : a \le i < b\}$.*

We look for elements f of V_τ which are symmetric or antisymmetric for a group $\mathcal{S}_{[a,b]}$, or the equivalent properties: $s_if = f$, respectively, $s_if = -f$, for $a \le i < b$. Roughly, start with some v_T and analyze $\sum_{w\in\mathcal{S}_{[a,b]}} wv_T$, or $\sum_{w\in\mathcal{S}_{[a,b]}} \mathrm{sgn}(w)\ wv_T$, expanded in the basis $\{v_S : S \in Y(\tau)\}$.

Definition 4. *For $T \in Y(\tau)$ and a subgroup H of $\mathcal{S}_N$ let $V_T(H) = \mathrm{span}\{wv_T : w \in H\}$ and let $Y(T;H) = \{T' \in Y(\tau) : v_{T'} \in V_T(H)\}$.*

In the case $H = \mathcal{S}_{[a,b]}$ there are two extremal elements of $Y(T;H)$, namely T_0 with the property $\mathrm{cm}(i,T_0) \ge \mathrm{cm}(i+1,T_0)$ for $a \le i < b$, and T_1 with the property $\mathrm{rw}(i,T_0) \ge \mathrm{rw}(i+1,T_0)$ (it is possible that $T_0 = T_1$). To produce T_0 one applies a sequence of transformations of type (4) (in Prop. 1) (type (3) for T_1). If $\mathrm{cm}(i_1,T) = \mathrm{cm}(i_2,T)$ for some $i_1, i_2 \in [a,b]$ (suppose $i_1 > i_2$ then any entry j in this column of T between i_1 and i_2 has to satisfy $i_1 > j > i_2$) then T_0 has $\mathrm{cm}(i,T_0) = \mathrm{cm}(i+1,T_0)$ for some $i \in [a, b-1]$. Similarly if $\mathrm{rw}(i_1,T) = \mathrm{rw}(i_2,T)$ for some $i_1, i_2 \in [a,b]$ then T_1 has $\mathrm{rw}(i,T_1) = \mathrm{rw}(i+1,T_1)$ for some $i \in [a, b-1]$.

First consider the invariant (symmetric) situation. Corollary 1 and the properties of T_0 imply the following necessary condition for $V_T\left(\mathcal{S}_{[a,b]}\right)$ to contain a nontrivial $\mathcal{S}_{[a,b]}$-invariant.

Say T satisfies condition $[a,b]_{\mathrm{cm}}$ if $a \le i < j \le m$ implies $\mathrm{cm}(i,T) \ne \mathrm{cm}(j,T)$ (the entries $a, a+1, \ldots, b$ are in distinct columns of T). Fix some T satisfying this condition and consider the subspace $V_T\left(\mathcal{S}_{[a,b]}\right)$. Let $T_0 \in Y\left(T; \mathcal{S}_{[a,b]}\right)$ satisfy $\mathrm{cm}(i,T) > \mathrm{cm}(j,T)$ for $a \le i < j \le b$ (equality is ruled out by hypothesis). It is possible that i and $i+1$ are in the same row of T_0 for some $i \in [a,b]$ (in which case $\#Y\left(T; \mathcal{S}_{[a,b]}\right) < (b-a+1)! = \#\mathcal{S}_{[a,b]}$). For $a \le i < b$ we have $\mathrm{rw}(i,T_0) \le \mathrm{rw}(i+1,T_0)$, thus $a \le i < j \le b$ implies $c(j,T_0) - c(i,T_0) \le -2$ or $j = i+1$ and $\mathrm{rw}(i+1,T_0) = \mathrm{rw}(i,T_0)$; indeed suppose the latter condition does not hold then

if $j > i + 1$

$$\begin{aligned} c(j, T_0) - c(i, T_0) &= (\mathrm{cm}(j, T_0) - \mathrm{cm}(i, T_0)) + (\mathrm{rw}(i, T_0) - \mathrm{rw}(j, T_0)) \\ &\leq \mathrm{cm}(j, T_0) - \mathrm{cm}(i, T_0) \leq i - j \leq -2, \end{aligned}$$

or $j = i + 1$ and

$$\begin{aligned} c(i+1, T_0) - c(i, T_0) &= (\mathrm{cm}(i+1, T_0) - \mathrm{cm}(i, T_0)) + (\mathrm{rw}(i, T_0) - \mathrm{rw}(i+1, T_0)) \\ &\leq -1 - 1 = -2. \end{aligned}$$

Definition 5. *Suppose $T \in Y(\tau)$ satisfies condition $[a, b]_{\mathrm{cm}}$ then let*

$$P_0(T; a, b) = \prod_{a \leq i < j \leq b,\ \mathrm{cm}(i,T) < \mathrm{cm}(j,T)} \frac{c(j, T) - c(i, T)}{1 + c(j, T) - c(i, T)}.$$

The denominator can not vanish, for suppose $i < j, \mathrm{cm}(i, T) < \mathrm{cm}(j, T)$, and $T(i) = T_0(i_1), T(j) = T_0(i_2)$ with $i_1 < i_2$ (this follows from $\mathrm{cm}(i_2, T) < \mathrm{cm}(i_1, T)$) then $c(i, T) - c(j, T) = c(i_2, T_0) - c(i_1, T_0) \leq -2$, and $\mathrm{rw}(i, T) = \mathrm{rw}(i_2, T_0) \neq \mathrm{rw}(j, T) = \mathrm{rw}(i_1, T_0)$. For notational convenience we use the fact $Y\left(T; \mathcal{S}_{[a,b]}\right) = Y\left(T_0; \mathcal{S}_{[a,b]}\right)$ (and let T be variable, henceforth).

Proposition 3. *Let $f = \sum_{T \in Y\left(T_0; \mathcal{S}_{[a,b]}\right)} P_0(T; a, b) v_T$ then $wf = f$ for all $w \in \mathcal{S}_{[a,b]}$.*

Proof. Suppose $a \leq i < b$ then let $A = \left\{T \in Y\left(T_0; \mathcal{S}_{[a,b]}\right) : \mathrm{rw}(i, T) = \mathrm{rw}(i+1, T)\right\}$ and $B = \left\{T \in Y\left(T_0; \mathcal{S}_{[a,b]}\right) : \mathrm{rw}(i, T) < \mathrm{rw}(i+1, T)\right\}$. Then

$$f = \sum_{T \in A} P_0(T; a, b) v_T + \sum_{T \in B} \left(P_0(T; a, b) v_T + P_0(s_i T; a, b) v_{s_i T}\right).$$

Fix $T \in B$ and compute $P_0(T; a, b) / P_0(s_i T; a, b)$ using Lemma 1; set $g_{mn}(T) = 1$ if $\mathrm{cm}(m, T) \geq \mathrm{cm}(n, T)$ and $g_{mn}(T) = \frac{c(n,T) - c(m,T)}{1 + c(n,T) - c(m,T)}$ if $\mathrm{cm}(m, T) < \mathrm{cm}(n, T)$. Then $g_{i,i+1}(T) = 1$ and $g_{i,i+1}(s_i T) = \frac{c(i+1, s_i T) - c(i, s_i T)}{1 + c(i+1, s_i T) - c(i, s_i T)} = \frac{1}{1 - b_i(s_i T)} = \frac{1}{1 + b_i(T)}$. Thus $P_0(T; a, b) / P_0(s_i T; a, b) = 1 + b_i(T)$ and $s_i f = f$ by Proposition 1. □

Corollary 2. *Let $n_0 = \#\left\{w \in \mathcal{S}_{[a,b]} : w v_{T_0} = v_{T_0}\right\}$, then*

$$\|f\|_0^2 = \frac{(b-a)!}{n_0} P_0(T_1; a, b) \|v_{T_0}\|_0^2.$$

Proof. If $T, T' \in Y(\tau)$ and T' is obtained from T by a sequence of steps of type (3) in Proposition 1 then $T' = wT$ for some $w \in \mathcal{S}_N$ and $v_{T'} = w v_T + \sum_j b_j v_{S_j}$, where $b_j \in \mathbb{Q}$ and $[S_1 = T, S_2, \ldots]$ is the list of intermediate steps. Let $f_1 = \sum\limits_{w \in \mathcal{S}_{[a,b]}} w v_{T_0}$ thus $f_1 = cf$ for some constant c. In the expansion of f_1 in the basis $\left\{v_T : T \in Y\left(T_0; \mathcal{S}_{[a,b]}\right)\right\}$ the coefficient of v_{T_1} is n_0, because T_0, T_1 have the property described above and v_{T_1} is extremal in $Y\left(T_0; \mathcal{S}_{[a,b]}\right)$ (heuristically the "bubble sort" is used; first apply $(b-1, b)(b-2, b-1)\ldots(a, a+1)$ to T_0; this moves b to the column with highest possible number; then repeat the process with $\mathcal{S}_{[a,b-1]}$, or $\mathcal{S}_{[a,b-k]}$ if $b-k+1, \ldots, b$ are now in the same row, and so on). The

coefficient of v_{T_1} in f is $P_0\left(T_1;a,b\right)$. Thus $c=\frac{n_0}{P_0(T_1;a,b)}$in f. Finally

$$\begin{aligned}\langle f,f\rangle_0 &= \frac{1}{c}\langle f_1,f\rangle_0 = \frac{1}{c}\sum_{w\in\mathcal{S}_{[a,b]}}\langle wv_{T_0},f\rangle_0\\ &= \frac{(b-a)!}{c}\langle v_{T_0},f\rangle_0 = \frac{(b-a)!}{c}\langle v_{T_0},v_{T_0}\rangle.\end{aligned}$$

This completes the proof. □

It is straightforward to extend these methods to the case $H=\mathcal{S}_{[a_1,b_1]}\times\mathcal{S}_{[a_2,b_2]}\times\ldots\mathcal{S}_{[a_n,b_n]}$ where $1\leq a_1<b_1<a_2<b_2<\ldots<a_n<b_n\leq N$. This requires a tableau $T_0\in Y(\tau)$ satisfying condition $[a_i,b_i]_{\mathrm{cm}}$ for $1\leq i\leq n$. Then

$$f=\sum_{T\in Y(T_0;H)}\prod_{i=1}^{n}P_0\left(T;a_i,b_i\right)v_T$$

is the unique H-invariant element of $V_{T_0}(H)$.

We turn to the problem of antisymmetric vectors in $V_T(H)$. The previous arguments transfer almost directly by transposing tableaux and inserting minus signs at appropriate places.

Say T satisfies condition $[a,b]_{\mathrm{rw}}$ if $a\leq i<j\leq m$ implies $\mathrm{rw}(i,T)\neq\mathrm{rw}(j,T)$ (the entries $a,a+1,\ldots,b$ are in distinct rows of T). Fix some T satisfying this condition and consider the subspace $V_T\left(\mathcal{S}_{[a,b]}\right)$. Let $T_0\in Y\left(T;\mathcal{S}_{[a,b]}\right)$ satisfy $\mathrm{cm}(i,T)\geq\mathrm{cm}(j,T)$ for $a\leq i<j\leq b$.

Definition 6. *Suppose $T\in Y(\tau)$ satisfies condition $[a,b]_{\mathrm{cm}}$ then let*

$$P_1(T;a,b)=\prod_{a\leq i<j\leq b,\ \mathrm{cm}(i,T)<\mathrm{cm}(j,T)}\frac{c(j,T)-c(i,T)}{1-c(j,T)+c(i,T)}.$$

As before we use the basic set $Y\left(T_0;\mathcal{S}_{[a,b]}\right)$ to produce an anti-symmetric vector. Note $P_1(T_0;a,b)=1$.

Proposition 4. *Let $f=\sum_{T\in Y\left(T_0;\mathcal{S}_{[a,b]}\right)}P_1(T;a,b)\,v_T$ then $s_if=-f$ for $a\leq i<b$ and $wf=$sgn$(w)\,f$ for all $w\in\mathcal{S}_{[a,b]}$. Let $n_0=\#\left\{w\in\mathcal{S}_{[a,b]}:wv_{T_0}=\pm v_{T_0}\right\}$ then $\|f\|^2=\frac{(b-a)!}{n_0}\left|P_1(T_1;a,b)\right|\ \|v_{T_0}\|^2$.*

Proof. Suppose $a\leq i<b$ then let $A=\left\{T\in Y\left(T_0;\mathcal{S}_{[a,b]}\right):\mathrm{cm}(i,T)=\mathrm{cm}(i+1,T)\right\}$ and $B=\left\{T\in Y\left(T_0;\mathcal{S}_{[a,b]}\right):\mathrm{rw}(i,T)<\mathrm{rw}(i+1,T)\right\}$ (also $\mathrm{cm}(i,T)>\mathrm{cm}(i+1,T)$ for $T\in B$); $T\in A$ implies $s_iv_T=-v_T$. Then

$$f=\sum_{T\in A}P_1(T;a,b)\,v_T+\sum_{T\in B}\left(P_1(T;a,b)\,v_T+P_1(s_iT;a,b)\,v_{s_iT}\right).$$

Fix $T\in B$ and compute $P_1(T;a,b)/P_1(s_iT;a,b)$ using Lemma 1; set $g_{mn}(T)=1$ if $\mathrm{cm}(m,T)\geq\mathrm{cm}(n,T)$ and $g_{mn}(T)=\frac{c(n,T)-c(m,T)}{1-c(n,T)+c(m,T)}$ if $\mathrm{cm}(m,T)<\mathrm{cm}(n,T)$. Then $g_{i,i+1}(T)=1$ and $g_{i,i+1}(s_iT)=\frac{c(i+1,s_iT)-c(i,s_iT)}{1-c(i+1,s_iT)+c(i,s_iT)}=\frac{1}{-1-b_i(s_iT)}=\frac{1}{b_i(T)-1}$. Thus $P_0(T;a,b)/P_0(s_iT;a,b)=b_i(T)-1$ and $s_if=-f$ by Proposition 1. The norm formula follows from the proof of Corollary 2 with some small modifications to take care of sign-changes. □

There are corresponding statements for $H = \mathcal{S}_{[a_1,b_1]} \times \mathcal{S}_{[a_2,b_2]} \times \ldots \mathcal{S}_{[a_n,b_n]}$, using disjoint intervals. The branching theorem for the restriction of irreducible representations of $\mathcal{S}_N$ to those of the parabolic subgroups (like H) implicitly appears in the previous discussion, in connection with the conditions $[a,b]_{cm}$ and $[a,b]_{rw}$.

3. Dunkl operators

Let κ be a transcendental (formal parameter) and set $\mathbb{F} = \mathbb{Q}(\kappa)$. Consider the space $\mathcal{P} \otimes V_\tau = \operatorname{span}_{\mathbb{F}} \left\{ x^\alpha v_T : \alpha \in \mathbb{N}_0^N, T \in Y(\tau) \right\}$, polynomials $p(x)$ on $\mathbb{R}^N$ with values in V_τ. The space is an $\mathcal{S}_N$-module with the action $w(x^\alpha v_T) = x^{w\alpha}(wv_T)$ for $w \in \mathcal{S}_N$, extended to all of $\mathcal{P} \otimes V_\tau$ by linearity. For $p \in \mathcal{P}$ and $u \in V_\tau$ and $1 \le i \le N$ let

$$\mathcal{D}_i (p(x) u) := \frac{\partial}{\partial x_i} p(x) u + \kappa \sum_{j=1, j\neq i}^{N} \frac{p(x) - p(x(i,j))}{x_i - x_j} (i,j) u. \tag{3.1}$$

The definition is extended to $\mathcal{P} \otimes V_\tau$ by linearity. Then $\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i$ for $1 \le i,j \le N$. The proof is a straightforward adaptation of the original proof for scalar polynomials $p(x)$ (see [2, Ch. 4]). There are important commutators (appearing in the definition of the rational Cherednik algebra, the algebra generated by $\Bbbk\mathcal{S}_N$ and $\{x_i, \mathcal{D}_i : i \in [1,N]\}$):

$$\begin{aligned} \mathcal{D}_i x_j - x_j \mathcal{D}_i &= -\kappa (i,j), i \neq j \\ \mathcal{D}_i x_i - x_i \mathcal{D}_i &= 1 + \kappa \sum_{j \neq i} (i,j). \end{aligned} \tag{3.2}$$

Definition 7. *The space $\mathcal{P} \otimes V_\tau$ equipped with the action of $\mathbb{F}\mathcal{S}_N$ and $\{x_i, \mathcal{D}_i : i \in [1,N]\}$ is a standard module of the rational Cherednik algebra and is denoted by $M(\tau)$.*

The representation theory of rational Cherednik algebras is described in the survey [9] by Rouquier. For $p(x) \in \mathcal{P} \otimes V_\tau$ set

$$\mathcal{U}_i p(x) = \mathcal{D}_i (x_i p(x)) - \kappa \sum_{j=1}^{i-1} (i,j) p(x), 1 \le i \le N.$$

The operators $\mathcal{U}_i$ also commute pairwise. They have a triangularity property (a special case of a result of Griffeth [5] for the complex reflection groups $G(p,r,N)$). There is an important function on compositions:

Definition 8. *For $\alpha \in \mathbb{N}_0^N$ and $1 \le i \le N$ let $r(\alpha, i) := \#\{j : \alpha_j > \alpha_i\} + \#\{j : 1 \le j \le i, \alpha_j = \alpha_i\}$ be the rank function.*

A consequence of the definition is that $r(\alpha,i) < r(\alpha,j)$ is equivalent to $\alpha_i > \alpha_j$, or $\alpha_i = \alpha_j$ and $i < j$. For any α the function $i \mapsto r(\alpha,i)$ is one-to-one on $\{1,2,\ldots,N\}$. Let w_α denote the inverse function, thus $r(\alpha, w_\alpha(i)) = i$. Further α is a partition if and only if $r(\alpha,i) = i$ for all i. In general $\left(w_\alpha^{-1}\alpha\right)_i = \alpha_{w_\alpha(i)}$ for $1 \le i \le N$, and thus $w_\alpha^{-1}\alpha$ is a partition, denoted by α^+. The order on compositions is derived from the dominance order.

Definition 9. *For $\alpha, \beta \in \mathbb{N}_0^N$ the partial order $\alpha \succ \beta$ (α dominates β) means that $\alpha \neq \beta$ and $\sum_{i=1}^{j} \alpha_i \ge \sum_{i=1}^{j} \beta_i$ for $1 \le j \le N$; and $\alpha \rhd \beta$ means that $|\alpha| = |\beta|$ and either $\alpha^+ \succ \beta^+$ or $\alpha^+ = \beta^+$ and $\alpha \succ \beta$.*

There are some results useful in analyzing $\mathcal{U}_i x^\alpha u$. Let $\varepsilon(i)$ be the ith standard basis vector in $\mathbb{N}_0^N$, for $1 \le i \le N$. By [2, Lemma 8.2.3] the following hold for $\alpha \in \mathbb{N}_0^N$:

(1) if $\alpha_i > \alpha_j$ and $i < j$ then $(i,j)\,\alpha \lhd \alpha$;
(2) $\alpha^+ \unrhd \alpha$;
(3) if $1 \le s < \alpha_i - \alpha_j$ then $\alpha^+ \rhd (\alpha - s(\varepsilon(i) - \varepsilon(j)))^+$.

The following is a consequence of these relations and an easy computation.

Lemma 2. *For $\alpha \in \mathbb{N}_0^N$ and $i \ne j$ let $B_{ij}x^\alpha = (x_i x^\alpha - x_j (i,j)\, x^\alpha)/(x_i - x_j)$, then*
(1) if $\alpha_i = \alpha_j$ then $B_{ij}x^\alpha = x^\alpha$;
(2) if $\alpha_i > \alpha_j$ then $B_{ij}x^\alpha = x^\alpha + (i,j)\, x^\alpha + \sum_{s=1}^{\alpha_i-\alpha_j-1} x^{\alpha - s(\varepsilon(i)-\varepsilon(j))}$ and $\alpha^+ \rhd (\alpha - s(\varepsilon(i) - \varepsilon(j)))^+$ for $1 \le s \le \alpha_i - \alpha_j - 1$;
(3) if $\alpha_i < \alpha_j$ then $B_{ij}x^\alpha = -\sum_{s=1}^{\alpha_j-\alpha_i-1} x^{\alpha - s(\varepsilon(j)-\varepsilon(i))}$ and $\alpha^+ \rhd (\alpha - s(\varepsilon(j) - \varepsilon(i)))^+$ for $1 \le s \le \alpha_j - \alpha_i - 1$.

The following Proposition can be elegantly stated in terms of conjugates of Jucys-Murphy elements. Recall the conjugation relation $w(i,j)\, w^{-1} = (w(i), w(j))$.

Definition 10. *For $\alpha \in \mathbb{N}_0^N$ and $1 \le i \le N$ let $\omega_i^\alpha := w_\alpha \omega_{r(\alpha,i)} w_\alpha^{-1}$, where w_α is the inverse of $r(\alpha,\cdot)$. Equivalently $\omega_i^\alpha = \sum \{(i,j) : r(\alpha,j) > r(\alpha,i)\}$.*

To justify the second equation observe that

$$w_\alpha \omega_{r(\alpha,i)} w_\alpha^{-1} = \sum_{r(\alpha,i)<j} (w_\alpha(r(\alpha,i)), w_\alpha(j)) = \sum_{r(\alpha,i)<j} (i, w_\alpha(j))$$

and $r(\alpha, w_\alpha(j)) = j$.

Proposition 5. *Suppose $\alpha \in \mathbb{N}_0^N$, $u \in V_\tau$ and $1 \le i \le N$ then*

$$\mathcal{U}_i x^\alpha u = x^\alpha \left[(\alpha_i + 1)\, u + \kappa \omega_i^\alpha u\right] + \kappa \sum_{\beta \lhd \alpha} x^\beta u_\beta,$$

where each $u_\beta = 0$ or $\pm (i,j)\, u$ for some j.

Proof. Let q_α denote elements of span $\{x^\beta : \beta \lhd \alpha\}$. In the case $1 \le j < i$ the coefficient of $\kappa(i,j)\, u$ is $B_{ij}x^\alpha - (i,j)\, x^\alpha$ which equals (1) 0 if $\alpha_i = \alpha_j$, (2) $x^\alpha + q_\alpha$ if $\alpha_i > \alpha_j$,(3) $-((i,j)\, x^\alpha + q_\alpha)$ if $\alpha_j > \alpha_i$, so that $(i,j)\,\alpha \lhd \alpha$. In the case $i < j \le N$ the coefficient of $\kappa(i,j)\, u$ is $B_{ij}x^\alpha$ which equals (1) x^α if $\alpha_i = \alpha_j$, (2) $x^\alpha + (i,j)\, x^\alpha + q_\alpha$ if $\alpha_i > \alpha_j$, so that $(i,j)\,\alpha \lhd \alpha$, (3) q_α if $\alpha_i < \alpha_j$. Thus $\kappa x^\alpha (i,j)\, u$ appears in $\mathcal{U}_i x^\alpha u$ exactly when $\alpha_i > \alpha_j$ or $\alpha_i = \alpha_j$ and $j > i$, that is, $r(\alpha,j) > r(\alpha,i)$. □

Following Griffeth we define an order on the pairs $\{(\alpha,u) : \alpha \in \mathbb{N}_0^N\}$: $(\alpha,u_1) \rhd (\beta,u_2)$ means that $\alpha \rhd \beta$. For this order the leading term of $\mathcal{U}_i x^\alpha u$ is $x^\alpha (\alpha_i + 1 + \kappa\omega_i^\alpha)\, u$.

4. Nonsymmetric Jack polynomials

This section presents the structure of the simultaneous eigenvectors of $\{\mathcal{U}_i : 1 \le i \le N\}$ in $M(\tau)$. These are vector-valued generalizations of the nonsymmetric Jack polynomials (see [2, Ch. 8]). The operators $\mathcal{U}_i$ are self-adjoint with respect to the contravariant form, which is described as follows:

The contravariant form $\langle\cdot,\cdot\rangle$ on $M(\tau)$ is the canonical symmetric $\mathcal{S}_N$-invariant bilinear form, extending the form $\langle\cdot,\cdot\rangle_0$ on V_τ, : such that

$$\langle x_i f, g\rangle = \langle f, \mathcal{D}_i g\rangle, i\in[1,N], f,g\in M(\tau).$$

An existence proof can be based on the operator $\sum_{i=1}^{N} x_i\mathcal{D}_i + \kappa\sum_{1\le i<j\le N}(i,j)$ and induction. The important properties of the form are:

1. if $f\in\mathcal{P}_m\otimes V_\tau, g\in\mathcal{P}_n\otimes V_\tau$ and $m\neq n$ then $\langle f,g\rangle = 0$;
2. if $w\in\mathcal{S}_N$ then $\langle wf, wg\rangle = \langle f,g\rangle$ for all $f,g\in M(\tau)$, if $1\le i<j\le N$ then $\langle (i,j)f, g\rangle = \langle f,(i,j)g\rangle$;
3. if $1\le i\le N$ and $f,g\in M(\tau)$ then $\langle \mathcal{D}_i x_i f, g\rangle = \langle f, \mathcal{D}_i x_i g\rangle$.

We use $\|f\|^2$ to denote $\langle f,f\rangle$ although the form may not be positive-definite . For a specific value $\kappa\in\mathbb{Q}$ the kernel of the form, that is, $\{f:\langle g,f\rangle=0,\forall g\in M(\tau)\}$, is called the *radical* of $M(\tau)$ and denoted $J_\kappa(\tau)$, and the quotient module $M(\tau)/J_\kappa(\tau)$ is denoted $L_\kappa(\tau)$. Values of κ such that $J_\kappa(\tau)\neq(0)$ are called *singular* values.

We note that if $\lambda\in\mathbb{N}_0^{N,+}$ then the leading term in $\mathcal{U}_i x^\lambda u$ is $x^\lambda(\lambda_i+1+\kappa\omega_i)u$; this suggests that eigenvectors of ω_i have good properties under the action of $\mathcal{U}_i$. For compositions the coordinates have to be appropriately permuted. From (5) we see that for $T\in Y(\tau)$ and $\alpha\in\mathbb{N}_0^N$ the leading term in $\mathcal{U}_i x^\alpha w_\alpha v_T$ is $(\alpha_i+1+\kappa c(r(\alpha,i),T))x^\alpha w_\alpha v_T$, because $\omega_i^\alpha w_\alpha v_T = w_\alpha\omega_{r(\alpha,i)}v_T = c(r(\alpha,i),T)w_\alpha v_T$. For any $n\in\mathbb{N}_0$ the set $\{x^\alpha w_\alpha v_T:\alpha\in\mathbb{N}_0^N,|\alpha|=n,T\in Y(\tau)\}$ is a basis of $M_n(\tau):=\mathcal{P}_n\otimes V_\tau$ on which the operators $\mathcal{U}_i$ act in a triangular manner (with respect to $\rhd$). For $\alpha\in\mathbb{N}_0^N, T\in Y(\tau)$, let

$$\xi_i(\alpha,T) = \alpha_i+1+\kappa c(r(\alpha,i),T), 1\le i\le N.$$

For any $\beta\neq\alpha$ and $|\beta|=|\alpha|$ there is at least one i such that $\alpha_i\neq 0$ and $\alpha_i\neq\beta_i$ thus $\xi_i(\alpha,T)\neq\xi_i(\beta,T')$ for any $T,T'\in Y(\tau)$ (and generic κ). (The restriction to $\alpha_i\neq 0$ is needed in the next section; if $|\alpha|=|\beta|$ and $\alpha_i\neq 0$ implies $\alpha_i=\beta_i$ then $\alpha=\beta$.) Thus there exists a basis of simultaneous eigenvectors of $\{\mathcal{U}_i:1\le i\le N\}$. The following is the specialization to $\mathcal{S}_N$ of Griffeth's construction [5, Theorem 5.2] of nonsymmetric Jack polynomials.

Proposition 6. *For $\alpha\in\mathbb{N}_0^N, T\in Y(\tau)$ there exists a unique element $\zeta_{\alpha,T}$ of $M(\tau)$ such that $\mathcal{U}_i\zeta_{\alpha,T}=\xi_i(\alpha,T)\zeta_{\alpha,T}$ for $1\le i\le N$ and*

$$\zeta_{\alpha,T}(x) = x^\alpha w_\alpha v_T + \sum_{\beta\lhd\alpha} x^\beta g_{\beta\alpha},$$

where $g_{\beta\alpha}\in V_\tau$.

The existence of this set of simultaneous eigenvectors of $\{\mathcal{U}_i:1\le i\le N\}$ follows from the triangular property, the commutativity, and the separation properties of the eigenvalues $(\alpha,T)\mapsto[\xi_i(\alpha,T)]_{i=1}^N$.

Because each $\mathcal{U}_i$ is self-adjoint for $\langle\cdot,\cdot\rangle$ we have $\langle\zeta_{\alpha,T},\zeta_{\beta,T'}\rangle=0$ when $\alpha\neq\beta$ or $T\neq T'$.

We consider the action of $\mathcal{S}_N$ on the polynomials $\zeta_{\alpha,T}$. As usual there are explicit formulae for the action of $s_i=(i,i+1)$ based on the commutations $\mathcal{U}_j s_i = s_i\mathcal{U}_j$ for $j\neq i,i+1$ and $s_i\mathcal{U}_i s_i=\mathcal{U}_{i+1}+\kappa$. These are special cases of [5, Theorem 5.3], however we use the nonnormalized basis for V_τ rather than the orthonormal one used there (so coefficients in $\mathbb{Q}(\kappa)$ suffice). As in [5] let σ_i denote the formal

operator $s_i + \frac{\kappa}{\mathcal{U}_{i+1}-\mathcal{U}_i}$; suppose $f \in M(\tau)$ and $\mathcal{U}_j f = \lambda_j f$ for $1 \leq j \leq N$ (with $\lambda_j \in \mathbb{Q}(\kappa)$ and $\lambda_i \neq \lambda_{i+1}$) then $\mathcal{U}_j \sigma_i f = \lambda_j \sigma_i f$ for $j \neq i, i+1$ and $\mathcal{U}_i \sigma_i f = \lambda_{i+1} \sigma_i f$, $\mathcal{U}_{i+1} \sigma_i f = \lambda_i \sigma_i f$ (where $\sigma_i f = s_i f + \frac{\kappa}{\lambda_{i+1}-\lambda_i} f$. Specifically there are two main cases $\alpha_i \neq \alpha_{i+1}$ and $\alpha_i = \alpha_{i+1}$. For $\alpha \in \mathbb{N}_0^N$ and $T \in Y(\tau)$ let

$$\begin{aligned} b_i(\alpha,T) &= \frac{\kappa}{\xi_i(\alpha,T) - \xi_{i+1}(\alpha,T)} \\ &= \frac{\kappa}{\alpha_i - \alpha_{i+1} + \kappa\left(c\left(r(\alpha,i),T\right) - c\left(r(\alpha,i+1),T\right)\right)}. \end{aligned}$$

Proposition 7. *Suppose $\alpha \in \mathbb{N}_0^N$ and $\alpha_i > \alpha_{i+1}$ for some $i < N$. Then*

$$\begin{aligned} s_i \zeta_{\alpha,T} &= b_i(\alpha,T)\,\zeta_{\alpha,T} + \left(1 - b_i(\alpha,T)^2\right)\zeta_{s_i\alpha,T}, \\ s_i \zeta_{s_i\alpha,T} &= \zeta_{\alpha,T} - b_i(\alpha,T)\,\zeta_{s_i\alpha,T}; \\ \left\|\zeta_{\alpha,T}\right\|^2 &= \left(1 - b_i(\alpha,T)^2\right)\left\|\zeta_{s_i\alpha,T}\right\|^2. \end{aligned}$$

Proof. The condition $\alpha_i \neq \alpha_{i+1}$ implies $r(s_i\alpha, i) = r(\alpha, i+1)$ and $r(s_i\alpha, i+1) = r(\alpha, i)$, thus $\xi_i(s_i\alpha, T) = \xi_{i+1}(\alpha, T)$ and $\xi_{i+1}(s_i\alpha, T) = \xi_i(\alpha, T)$ (and $\xi_j(s_i\alpha, T) = \xi_j(\alpha, T)$ for $j \neq i, i+1$). Since the eigenvalues determine the eigenvectors uniquely we have that

$$\begin{aligned} s_i \zeta_{\alpha,T} - b_i(\alpha,T)\,\zeta_{\alpha,T} &= a\zeta_{s_i\alpha,T}, \\ s_i \zeta_{s_i\alpha,T} + b_i(\alpha,T)\,\zeta_{s_i\alpha,T} &= a'\zeta_{\alpha,T}, \end{aligned}$$

for some scalars a, a'. The fact that $s_i^2 = 1$ implies $aa' = 1 - b_i(\alpha,T)^2$. We show that $a' = 1$ by finding the leading term in $s_i\zeta_{s_i\alpha,T}$,namely $x^\alpha s_i w_{s_i\alpha} v_T$. It remains to show that $w_{s_i\alpha} = s_i w_\alpha$, that is, $r(s_i\alpha, s_i w_\alpha(j)) = j$ for all j. If $w_\alpha^{-1}(j) \neq i, i+1$ then $r(s_i\alpha, s_i w_\alpha(j)) = r(\alpha, w_\alpha(j)) = j$. If $w_\alpha^{-1}(j) = i$ then $r(s_i\alpha, s_i w_\alpha(j)) = r(s_i\alpha, i+1) = r(\alpha, i) = j$. The case $w_\alpha^{-1}(j) = i+1$ follows similarly. The second displayed equation shows that $\left\|s_i\zeta_{s_i\alpha,T}\right\|^2 = \left\|\zeta_{s_i\alpha,T}\right\|^2 = \left\|a'\zeta_{\alpha,T}\right\|^2 - b_i(\alpha,T)^2\left\|\zeta_{s_i\alpha,T}\right\|^2$. $\square$

Remark 1. *A necessary condition for the form $\langle\cdot,\cdot\rangle$ to be positive-definite now becomes apparent: $b_i(\alpha,T)^2 < 1$ for all i, α, T. The "trivial" cases are $\tau = (N)$ and $\tau = (1,\ldots,1)$ for which $\kappa > -\frac{1}{N}$ and $\kappa < \frac{1}{N}$ are necessary and sufficient, respectively. Otherwise let $h_\tau := \tau_1 + \ell(\tau) - 1$, the maximum hook-length of τ, then $-\frac{1}{h_\tau} < \kappa < \frac{1}{h_\tau}$ implies $b_i(\alpha,T)^2 < 1$ for all i, α, T. Note that $1 \leq i, j \leq N, T \in Y(\tau)$ implies $|c(i,T) - c(j,T)| \leq h_\tau - 1$.*

Etingof, Stoica and Griffeth [3, Thm. 5.5] found the complete description of the set of values of κ for which $L_\kappa(\tau)$ provides a unitary representation of the rational Cherednik algebra. We can find an expression for $\left\|\zeta_{\alpha,T}\right\|^2$ in terms of $\left\|\zeta_{\alpha^+,T}\right\|^2$, following the approach used in [2, Thm. 8.5.8].

Definition 11. *For $\alpha \in \mathbb{N}_0^N, T \in Y(\tau)$ and $\varepsilon = \pm$ let*

$$\mathcal{E}_\varepsilon(\alpha,T) = \prod_{1 \leq i < j \leq N, \alpha_i < \alpha_j} \left(1 + \frac{\varepsilon\kappa}{\alpha_j - \alpha_i + \kappa\left(c\left(r(\alpha,j),T\right) - c\left(r(\alpha,i),T\right)\right)}\right),$$

and let $\mathcal{E}_2(\alpha,T) = \mathcal{E}_+(\alpha,T)\,\mathcal{E}_-(\alpha,T)$.

Definition 12. *For $\alpha \in \mathbb{N}_0^N$ let* $\operatorname{inv}(\alpha) = \#\{(i,j) : 1 \leq i < j \leq N, \alpha_i < \alpha_j\}$.

Proposition 8. *Suppose $\alpha \in \mathbb{N}_0^N, T \in Y(\tau), \varepsilon = \pm$ and $\alpha_{i+1} > \alpha_i$ for some $i \in [1, N-1]$ then $\mathcal{E}_\varepsilon(s_i\alpha, T)/\mathcal{E}_\varepsilon(\alpha, T) = 1 + \varepsilon b_i(\alpha, T)$.*

Proof. Using an argument similar to that of Lemma 1 we have $\mathcal{E}_\varepsilon(s_i\alpha, T)/\mathcal{E}_\varepsilon(\alpha, T) = 1 + \frac{\varepsilon\kappa}{(s_i\alpha)_{i+1} - (s_i\alpha)_i + \kappa(c(r(s_i\alpha, i+1), T) - c(r(s_i\alpha, i), T))} = 1 + \varepsilon b_i(\alpha, T)$, because $r(s_i\alpha, i+1) = r(\alpha, i)$ and $r(s_i\alpha, i) = r(\alpha, i+1)$. □

Corollary 3. *Suppose $\alpha \in \mathbb{N}_0^N, T \in Y(\tau)$ then $\|\zeta_{\alpha,T}\|^2 = \mathcal{E}_2(\alpha, T)^{-1} \left\|\zeta_{\alpha^+,T}\right\|^2$.*

Proof. Argue by induction on $\operatorname{inv}(\alpha)$. If the formula is valid for some α with $\alpha_i > \alpha_{i+1}$ then by Proposition 7

$$\begin{aligned}
\|\zeta_{s_i\alpha,T}\|^2 &= \left(1 - b_i(\alpha, T)^2\right)^{-1} \|\zeta_{\alpha,T}\|^2 \\
&= \left(1 - b_i(\alpha, T)^2\right)^{-1} \mathcal{E}_2(\alpha, T)^{-1} \left\|\zeta_{\alpha^+,T}\right\|^2 \\
&= \mathcal{E}_2(s_i\alpha, T)^{-1} \left\|\zeta_{\alpha^+,T}\right\|^2.
\end{aligned}$$

This completes the induction. □

Consider the case $\alpha_i = \alpha_{i+1}$ and let $I = r(\alpha, i)$ so that $r(\alpha, i+1) = I + 1$ and $b_i(\alpha, T) = (c(I, T) - c(I+1, T))^{-1} = b_I(T)$ (see Proposition 1). Furthermore $s_i w_\alpha = w_\alpha\left(w_\alpha^{-1}(i), w_\alpha^{-1}(i+1)\right) = w_\alpha(I, I+1) = w_\alpha s_I$. The transformation properties depend on the positions of I and $I+1$ in T.

Proposition 9. *Suppose $\alpha \in \mathbb{N}_0^N, T \in Y(\tau)$ and $\alpha_i = \alpha_{i+1}$ for some $i < N$. For $I = r(\alpha, T)$ the following hold:*
1) if $b_I(T) = 1$ then $s_i\zeta_{\alpha,T} = \zeta_{\alpha,T}$,
2) if $b_I(T) = -1$ then $s_i\zeta_{\alpha,T} = -\zeta_{\alpha,T}$,
3) if $-\frac{1}{2} \le b_I(T) < 0$ then $s_i\zeta_{\alpha,T} = b_I(T)\zeta_{\alpha,T} + \left(1 - b_I(T)^2\right)\zeta_{\alpha,s_IT}$,
4) if $0 < b_I(T) \le \frac{1}{2}$ then $s_i\zeta_{\alpha,T} = b_I(T)\zeta_{\alpha,T} + \zeta_{\alpha,s_IT}$.

Proof. It suffices to consider the action of s_i on the leading term of $\zeta_{\alpha,T}$. Indeed $s_i x^\alpha w_\alpha v_T = x^\alpha w_\alpha(s_I v_T)$ and we use the equations from Proposition 1. □

Note that in case (3) $\|\zeta_{\alpha,T}\|^2 = \left(1 - b_I(T)^2\right)\|\zeta_{\alpha,s_IT}\|^2$ (and the reciprocal in case (4)). There is a raising operator involving a cyclic shift and multiplication by x_N. From the commutators 3.2 we obtain:

$$\begin{aligned}
\mathcal{U}_i x_N f &= x_N(\mathcal{U}_i - \kappa(i, N))f,\ 1 \le i < N, \\
\mathcal{U}_N x_N f &= x_N(1 + \mathcal{D}_N x_N)f.
\end{aligned}$$

Let $\theta_N = s_1 s_2 \ldots s_{N-1}$ thus $\theta_N(N) = 1$ and $\theta_N(i) = i+1$ for $1 \le i < N$ (a cyclic shift). Then

$$\begin{aligned}
\mathcal{U}_i x_N f &= x_N\left(\theta_N^{-1}\mathcal{U}_{i+1}\theta_N\right)f,\ 1 \le i < N, \\
\mathcal{U}_N x_N f &= x_N\left(1 + \theta_N^{-1}\mathcal{U}_1\theta_N\right)f.
\end{aligned}$$

If f satisfies $\mathcal{U}_i f = \lambda_i f$ for $1 \le i \le N$ then $\mathcal{U}_i\left(x_N\theta_N^{-1}f\right) = \lambda_{i+1}\left(x_N\theta_N^{-1}f\right)$ for $1 \le i < N$ and $\mathcal{U}_N\left(x_N\theta_N^{-1}f\right) = (\lambda_1 + 1)\left(x_N\theta_N^{-1}f\right)$. For $\alpha \in \mathbb{N}_0^N$ let $\phi(\alpha) := (\alpha_2, \alpha_3, \ldots, \alpha_N, \alpha_1 + 1)$, then $x_N\theta_N^{-1}x^\alpha = x^{\phi(\alpha)}$.

Proposition 10. *Suppose $\alpha \in \mathbb{N}_0^N, T \in Y(\tau)$, then $\zeta_{\phi(\alpha),T} = x_N\theta_N^{-1}\zeta_{\alpha,T}$.*

Proof. By straightforward arguments it follows that $r(\phi(\alpha),i) = r(\alpha,i+1)$ for $1 \le i < N$ and $r(\phi(\alpha),N) = r(\alpha,1)$, that is, $r(\phi(\alpha),i) = r(\alpha,\theta_N(i))$ for all i. This is equivalent to $r\left(\phi(\alpha),\theta_N^{-1}(w_\alpha(j))\right) = r(\alpha,w_\alpha(j)) = j$ for all j, or $w_{\phi(\alpha)} = \theta_N^{-1}w_\alpha$. The leading term $x^\alpha w_\alpha v_T$ of $\zeta_{\alpha,T}$ is mapped to $x^{\phi(\alpha)}w_{\phi(\alpha)}v_T$ by $f \mapsto x_N\theta_N^{-1}f$. Note that $\mathcal{U}_i\zeta_{\phi(\alpha),T} = (\alpha_{i+1}+1+\kappa c(r(\phi(\alpha),i),T))\,\zeta_{\phi(\alpha),T}$ for $1 \le i < N$ and $\mathcal{U}_N\zeta_{\phi(\alpha),T} = (\alpha_1+2+\kappa c(r(\phi(\alpha),N),T))\,\zeta_{\phi(\alpha),T}$. Thus $x_N\theta_N^{-1}\zeta_{\alpha,T}$ and $\zeta_{\phi(\alpha),T}$ have the same eigenvalues for $\{\mathcal{U}_i\}$ and the same coefficient of $x^{\phi(\alpha)}$.Hence $x_N\theta_N^{-1}\zeta_{\alpha,T} = \zeta_{\phi(\alpha),T}$. □

Corollary 4. $\left\|\zeta_{\phi(\alpha),T}\right\|^2 = (\alpha_1+1+\kappa c(r(\alpha,1),T))\left\|\zeta_{\alpha,T}\right\|^2$.

Proof. Indeed $\left\|\zeta_{\phi(\alpha),T}\right\|^2 = \left\langle\theta_N^{-1}\zeta_{\alpha,T},\mathcal{D}_Nx_N\theta_N^{-1}\zeta_{\alpha,T}\right\rangle = \left\langle\theta_N^{-1}\zeta_{\alpha,T},\theta_N^{-1}\mathcal{D}_1x_1\zeta_{\alpha,T}\right\rangle$ $= \langle\zeta_{\alpha,T},\mathcal{U}_1\zeta_{\alpha,T}\rangle = \xi_1(\alpha,T)\left\|\zeta_{\alpha,T}\right\|^2$. □

Griffeth [5, Thm. 6.1] showed the following:

Theorem 2. *Suppose* $\lambda \in \mathbb{N}_0^{N,+}$ *and* $T \in Y(\tau)$ *then*

$$\left\|\zeta_{\lambda,T}\right\|^2 = \left\|v_T\right\|_0^2\prod_{i=1}^{N}(1+\kappa c(i,T))_{\lambda_i}\prod_{1\le i<j\le N}\prod_{l=1}^{\lambda_i-\lambda_j}\left(1-\frac{\kappa^2}{(l+\kappa(c(i,T)-c(j,T)))^2}\right).$$

Proof. Argue by induction. Suppose $\lambda_1 = \lambda_2 = \ldots = \lambda_m > \lambda_{m+1}$. Let

$$\begin{aligned}
\beta &= (\lambda_1,\ldots,\lambda_{m-1},\lambda_{m+1},\ldots,\lambda_N,\lambda_1),\\
\alpha &= (\lambda_1-1,\lambda_1,\ldots,\lambda_{m-1},\lambda_{m+1},\ldots,\lambda_N),\\
\mu &= (\lambda_1,\ldots,\lambda_{m-1},\lambda_1-1,\lambda_{m+1},\ldots,\lambda_N).
\end{aligned}$$

Thus $\beta = \phi(\alpha)$ and

$$\begin{aligned}
\left\|\zeta_{\beta,T}\right\|^2 &= (\lambda_1+\kappa c(m,T))\left\|\zeta_{\alpha,T}\right\|^2\\
&= (\lambda_1+\kappa c(m,T))\,\mathcal{E}_2(\alpha,T)^{-1}\left\|\zeta_{\mu,T}\right\|^2,\\
\left\|\zeta_{\lambda,T}\right\|^2 &= \mathcal{E}_2(\beta,T)\left\|\zeta_{\beta,T}\right\|^2.
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{E}_\varepsilon(\alpha,T) &= \prod_{j=2}^{m}\left(1+\frac{\varepsilon\kappa}{1+\kappa(c(j-1,T)-c(m,T))}\right),\\
\mathcal{E}_\varepsilon(\beta,T) &= \prod_{j=m+1}^{N}\left(1+\frac{\varepsilon\kappa}{\lambda_1-\lambda_j+\kappa(c(m,T)-c(j,T))}\right).
\end{aligned}$$

The validity of the formula for $\left\|\zeta_{\mu,T}\right\|^2$ thus implies the validity for $\left\|\zeta_{\lambda,T}\right\|^2$ (that is, the value of $\left\|\zeta_{\lambda,T}\right\|^2/\left\|\zeta_{\mu,T}\right\|^2$ from the formula agrees with $(\lambda_1+\kappa c(m,T))\dfrac{\mathcal{E}_2(\beta,T)}{\mathcal{E}_2(\alpha,T)}$). □

5. Symmetric and Antisymmetric Polynomials

We consider symmetric and antisymmetric linear combinations of $\{\zeta_{\alpha,T}\}$. Recall

$$b_i(\alpha,T)=\frac{\kappa}{\alpha_i-\alpha_{i+1}+\kappa\left(c\left(r\left(\alpha,i\right),T\right)-c\left(r\left(\alpha,i+1\right),T\right)\right)},$$
$$b_i(T)=\frac{1}{c\left(i,T\right)-c\left(i+1,T\right)},$$

for $\alpha\in\mathbb{N}_0^N, T\in Y(\tau), i\in[1,N-1]$. Here is a description of s_i-invariant polynomials for a given i:

(1) $\zeta_{\alpha,T}+(1-b_i(\alpha,T))\zeta_{s_i\alpha,T}$, for $\alpha_i>\alpha_{i+1}$;
(2) $(b_I(T)+1)\zeta_{\alpha,T}+\zeta_{\alpha,s_IT}$, for $\alpha_i=\alpha_{i+1}, I=r(\alpha,i)$ and $0<b_I(T)\le\frac{1}{2}$;
(3) $\zeta_{\alpha,T}$, for $\alpha_i=\alpha_{i+1}, I=r(\alpha,i)$ and $b_I(T)=1$ $(\mathrm{rw}(I,T)=\mathrm{rw}(I+1,T))$.

The antisymmetric polynomials for s_i $(s_if=-f)$ are

(1) $\zeta_{\alpha,T}-(1+b_i(\alpha,T))\zeta_{s_i\alpha,T}$, for $\alpha_i>\alpha_{i+1}$;
(2) $(b_I(T)-1)\zeta_{\alpha,T}+\zeta_{\alpha,s_IT}$, for $\alpha_i=\alpha_{i+1}, I=r(\alpha,i)$ and $0<b_I(T)\le\frac{1}{2}$;
(3) $\zeta_{\alpha,T}$, for $\alpha_i=\alpha_{i+1}, I=r(\alpha,i)$ and $b_I(T)=-1$ $(\mathrm{cm}(I,T)=\mathrm{cm}(I+1,T))$.

Now we construct invariants. In any orbit span $\{w\zeta_{\alpha,T}:w\in\mathcal{S}_N\}$ there must be a polynomial with leading term x^{α^+} so it suffices to consider the situation $\zeta_{\lambda,T}$ for partitions λ. We collect concepts for use in the sequel.

Notation 1. *For $\lambda\in\mathbb{N}_0^{N,+}$ let $W_\lambda=\{w\in\mathcal{S}_N:w\lambda=\lambda\}$, the stabilizer subgroup of λ. Thus*

$$W_\lambda=\mathcal{S}_{[a_1,b_1]}\times\mathcal{S}_{[a_2,b_2]}\times\ldots\mathcal{S}_{[a_n,b_n]},$$

where $1\le a_1<b_1<a_2<b_2<\ldots<a_n<b_n\le N$ (this means $\lambda_{a_1}=\lambda_{b_1}>\lambda_{b_1+1}$ and so forth). These intervals depend on λ but we will not incorporate this into the notation. Let $\lambda^R=(\lambda_N,\lambda_{N-1},\ldots,\lambda_1)\in\mathbb{N}_0^N$, the reverse of λ. The permutation w_{λ^R} is defined by $(w_{\lambda^R})^{-1}(i)=r\left(\lambda^R,i\right), i\in[1,N]$ (Definition 10).

Generally $w_{\lambda^R}\ne w_0$ where w_0 is the longest permutation given by $w_0(i)=N+1-i$ (example: $\lambda=(3,2,2,1)$ then $[w_{\lambda^R}(i)]_{i=1}^4=[4,2,3,1]$). The composition λ^R is the unique minimum for the order "$\succ$" on $\{\alpha:\alpha^+=\lambda\}$. For $\alpha^+=\lambda$ and $T\in Y(\tau)$ the leading term of $\zeta_{\alpha,T}$ is $x^\alpha w_\alpha v_T$ (where $w_\alpha^{-1}(i)=r(\alpha,i)$) and the minimality of λ^R implies that the expansion of $\zeta_{\lambda^R,T}$ has no term of the form $x^\alpha u$ with $u\in V_\tau$ when $\alpha\ne\lambda^R,\alpha^+=\lambda$. From the expressions in (2) above we see that the subgroup W_λ is an important part of the analysis. The formulae developed in Section 2 will be used.

Consider first the simplest case where v_T is W_λ-invariant, that is, each interval $[a_i,b_i]$ is contained in a row of T, for $1\le i\le n$ $(\mathrm{rw}(j,T)=\mathrm{rw}(b_i,T)$ for $a_i\le j\le b_i)$. Then $\sum_{w\in\mathcal{S}_N}w\zeta_{\lambda,T}=\sum_{\alpha^+=\lambda}A_\alpha\zeta_{\alpha,T}$ with coefficients to be determined.

Theorem 3. *Suppose $\lambda\in\mathbb{N}_0^{N,+}$ and $T\in Y(\tau)$ such that $w\in W_\lambda$ implies $wv_T=v_T$ then the polynomial $f_{\lambda,T}$ defined by*

$$f_{\lambda,T}=\sum_{\alpha^+=\lambda}\mathcal{E}_-(\alpha,T)\,\zeta_{\alpha,T},$$

is $\mathcal{S}_N$-invariant and

$$\left\|f_{\lambda,T}^s\right\|^2=\frac{N!}{\#W_\lambda}\frac{1}{\mathcal{E}_+(\lambda^R,T)}\left\|\zeta_{\lambda,T}\right\|^2.$$

Proof. Fix $i\in[1,N-1]$ and let

$$A=\left\{\alpha:\alpha^{+}=\lambda,\alpha_{i}=\alpha_{i+1}\right\},$$
$$B=\left\{\alpha:\alpha^{+}=\lambda,\alpha_{i}>\alpha_{i+1}\right\}.$$

Write

$$f_{\lambda,T}=\sum_{\alpha\in A}\mathcal{E}_{-}\left(\alpha,T\right)\zeta_{\alpha,T}+\sum_{\alpha\in B}\left(\mathcal{E}_{-}\left(\alpha,T\right)\zeta_{\alpha,T}+\mathcal{E}_{-}\left(s_{i}\alpha,T\right)\zeta_{s_{i}\alpha,T}\right).$$

Suppose $\alpha\in A$ then $r\left(i+1,\alpha\right)=r\left(i,\alpha\right)+1$ thus the values $r\left(i,\alpha\right)$ and $r\left(i+1,\alpha\right)$ belong to some interval $[a_j,b_j]$ (where $\mathcal{S}_{[a_j,b_j]}$ is a factor of W_λ) and are adjacent entries in some row of T, hence $s_i\zeta_{\alpha,T}=\zeta_{\alpha,T}$. Next let $\alpha\in B$ then the corresponding term in the sum is $\mathcal{E}_{-}\left(\alpha,T\right)\left(\zeta_{\alpha,T}+\frac{\mathcal{E}_{-}(s_i\alpha,T)}{\mathcal{E}_{-}(\alpha,T)}\zeta_{s_i\alpha,T}\right)$. Using the techniques of Lemma 1 we find that

$$\begin{aligned}\frac{\mathcal{E}_{-}\left(s_{i}\alpha,T\right)}{\mathcal{E}_{-}\left(\alpha,T\right)}&=1-\frac{\kappa}{\left(s_{i}\alpha\right)_{i+1}-\left(s_{i}\alpha\right)_{i}+\kappa\left(c\left(r\left(s_{i}\alpha,i+1\right),T\right)-c\left(r\left(s_{i}\alpha,i\right),T\right)\right)}\\&=1-\frac{\kappa}{\alpha_{i}-\alpha_{i+1}+\kappa\left(c\left(r\left(\alpha,i\right),T\right)-c\left(r\left(\alpha,i\right),T\right)\right)}=1-b_{i}\left(\alpha,T\right),\end{aligned}$$

and thus the term for α in the sum over B is s_i-invariant. Consider $g=\sum_{w\in\mathcal{S}_N}w\zeta_{\lambda^R,T}$; since g is $\mathcal{S}_N$-invariant it must equal a constant multiple γ of $f_{\lambda,T}$. To find γ consider the coefficients of $x^\lambda v_T$ in $f_{\lambda,T}$ and g. The leading term of $\zeta_{\lambda^R,T}$ is $x^{\lambda^R}w_{\lambda^R}\left(v_T\right)$. The coefficient in $f_{\lambda,T}$ is 1 (by definition of $\zeta_{\lambda,T}$). The term $x^\lambda v_T$ appears in $w\zeta_{\lambda^R,T}$ with coefficient 1 exactly when $w=w_1w_{\lambda^R}^{-1}$ for $w_1\in W_\lambda$. Thus $g=\left(\#W_\lambda\right)f_{\lambda,T}$ and

$$\begin{aligned}\left\Vert f_{\lambda,T}^{s}\right\Vert^{2}&=\frac{1}{\#W_{\lambda}}\left\langle g,f_{\lambda,T}\right\rangle=\frac{1}{\#W_{\lambda}}\sum_{w\in\mathcal{S}_{N}}\left\langle w\zeta_{\lambda^{R},T},f_{\lambda,T}^{s}\right\rangle\\&=\frac{N!}{\#W_{\lambda}}\left\langle\zeta_{\lambda^{R},T},f_{\lambda,T}^{s}\right\rangle=\frac{N!}{\#W_{\lambda}}\mathcal{E}_{-}\left(\lambda^{R},T\right)\left\Vert\zeta_{\lambda^{R},T}\right\Vert^{2}\\&=\frac{N!\mathcal{E}_{-}\left(\lambda^{R},T\right)}{\left(\#W_{\lambda}\right)\mathcal{E}_{2}\left(\lambda^{R},T\right)}\left\Vert\zeta_{\lambda,T}\right\Vert^{2}.\end{aligned}$$

This completes the proof. □

We turn to the corresponding antisymmetric function involving $\zeta_{\lambda,T}$ where v_T is antisymmetric for W_λ. That is each interval $[a_i,b_i]$ (appearing in W_λ) is contained in a column of T, $a_i\leq j\leq b_i$ implies $\mathrm{cm}\left(j,T\right)=\mathrm{cm}\left(b_i,T\right)$). The idea of the sign of a permutation needs to be adapted to allow for λ having some values equal to each other; the number of inversions $\mathrm{inv}\left(\alpha\right)$ fills this role.

Theorem 4. *Suppose $\lambda\in\mathbb{N}_0^{N,+}$ and $T\in Y\left(\tau\right)$ such that $s_i\in W_\lambda$ implies $s_iv_T=-v_T$ then the polynomial $f_{\lambda,T}^a$ defined by*

$$f_{\lambda,T}^{a}=\sum_{\alpha^{+}=\lambda}\left(-1\right)^{\mathrm{inv}\left(\alpha\right)}\mathcal{E}_{+}\left(\alpha,T\right)\zeta_{\alpha,T},$$

is $\mathcal{S}_N$-alternating, and

$$\left\Vert f_{\lambda,T}^{a}\right\Vert^{2}=\frac{N!}{\#W_{\lambda}}\frac{1}{\mathcal{E}_{-}\left(\lambda^{R},T\right)}\left\Vert\zeta_{\lambda,T}\right\Vert^{2}.$$

Proof. Fix $i \in [1, N-1]$ and let

$$A = \left\{\alpha : \alpha^+ = \lambda, \alpha_i = \alpha_{i+1}\right\},$$
$$B = \left\{\alpha : \alpha^+ = \lambda, \alpha_i > \alpha_{i+1}\right\}.$$

Note $\alpha \in B$ implies $\operatorname{inv}(s_i\alpha) = \operatorname{inv}(\alpha) + 1$. Write

$$f^a_{\lambda,T} = \sum_{\alpha\in A} (-1)^{\operatorname{inv}(\alpha)} \mathcal{E}_+(\alpha, T)\,\zeta_{\alpha,T} + \sum_{\alpha\in B} (-1)^{\operatorname{inv}(\alpha)} \left(\mathcal{E}_+(\alpha, T)\,\zeta_{\alpha,T} - \mathcal{E}_+(s_i\alpha, T)\,\zeta_{s_i\alpha,T}\right).$$

Suppose $\alpha \in A$ then $r(i+1, \alpha) = r(i, \alpha) + 1$ thus the values $r(i, \alpha)$ and $r(i+1, \alpha)$ belong to some interval $[a_j, b_j]$ (where $\mathcal{S}_{[a_j,b_j]}$ is a factor of W_λ) and are adjacent entries in some column of T, hence $s_i\zeta_{\alpha,T} = -\zeta_{\alpha,T}$. Next let $\alpha \in B$ then the corresponding term in the sum is $(-1)^{\operatorname{inv}(\alpha)} \mathcal{E}_+(\alpha, T)\left(\zeta_{\alpha,T} - \frac{\mathcal{E}_+(s_i\alpha,T)}{\mathcal{E}_+(\alpha,T)}\zeta_{s_i\alpha,T}\right)$, a scalar multiple of $\zeta_{\alpha,T} - (1 + b_i(\alpha, T))\,\zeta_{s_i\alpha,T}$, by an argument similar to the previous theorem. This term satisfies $s_if = -f$. Thus $s_if^a_{\lambda,T} = -f^a_{\lambda,T}$. Consider $g = \sum_{w\in\mathcal{S}_N}\operatorname{sgn}(w)\, w\zeta_{\lambda^R,T}$; since g is $\mathcal{S}_N$-alternating it must equal a constant multiple γ of $f_{\lambda,T}$. To find γ consider the coefficients of $x^\lambda v_T$ in $f_{\lambda,T}$ and g. The coefficient in $f_{\lambda,T}$ is 1 (by definition of $\zeta_{\lambda,T}$). The term $x^\lambda v_T$ appears in $w\zeta_{\lambda^R,T}$ exactly when $w = w_1 w_{\lambda^R}^{-1}$ for $w_1 \in W_\lambda$. Let $\varepsilon = \operatorname{sgn}(w_{\lambda^R}) = (-1)^{\operatorname{inv}(\lambda^R)}$, because the length of w_{λ^R} is $\operatorname{inv}(\lambda^R))$. Furthermore

$$\begin{aligned}
\operatorname{sgn}\left(w_1 w_{\lambda^R}^{-1}\right) w_1 w_{\lambda^R}^{-1}\zeta_{\lambda^R,T} &= \operatorname{sgn}\left(w_1 w_{\lambda^R}^{-1}\right) w_1 w_{\lambda^R}^{-1}\left(x^{\lambda^R} w_{\lambda^R} v_T\right) + h_1\\
&= \varepsilon \operatorname{sgn}(w_1)\, w_1\left(x^\lambda v_T\right) + h_2\\
&= \varepsilon x^\lambda v_T + h_2
\end{aligned}$$

where h_1 and h_2 are terms of lower order, that is, of the form $\sum_{\beta\lhd\lambda} x^\beta u_\beta$ with $u_\beta \in V_\tau$. Thus $g = \varepsilon\,(\#W_\lambda)\, f_{\lambda,T}$ and

$$\begin{aligned}
\left\|f^a_{\lambda,T}\right\|^2 &= \frac{\varepsilon}{\#W_\lambda}\langle g, f_{\lambda,T}\rangle = \frac{\varepsilon}{\#W_\lambda}\sum_{w\in\mathcal{S}_N}\operatorname{sgn}(w)\left\langle w\zeta_{\lambda^R,T}, f^s_{\lambda,T}\right\rangle\\
&= \frac{\varepsilon}{\#W_\lambda}\sum_{w\in\mathcal{S}_N}\operatorname{sgn}(w)\left\langle \zeta_{\lambda^R,T}, w^{-1}f^s_{\lambda,T}\right\rangle = \frac{\varepsilon N!}{\#W_\lambda}\left\langle \zeta_{\lambda^R,T}, f^s_{\lambda,T}\right\rangle\\
&= \frac{\varepsilon N!}{\#W_\lambda}(-1)^{\operatorname{inv}(\lambda^R)}\mathcal{E}_+\left(\lambda^R, T\right)\left\|\zeta_{\lambda^R,T}\right\|^2 = \frac{N!\mathcal{E}_+\left(\lambda^R, T\right)}{(\#W_\lambda)\,\mathcal{E}_2\left(\lambda^R, T\right)}\left\|\zeta_{\lambda,T}\right\|^2,
\end{aligned}$$

This completes the proof. ∎

For the general case we introduce the following:

Definition 13. *For $\lambda \in \mathbb{N}_0^{N,+}$ and $T \in Y(\tau)$ define the tableau $\lfloor\lambda, T\rfloor$ to be the assignment of $\lambda_1, \lambda_2, \ldots, \lambda_N$ to the nodes of the Ferrers diagram of τ so that the entry at $T(i)$ is $\lambda_i, i \in [1, N]$. Thus the entries of $\langle\lambda, T\rangle$ are weakly increasing ($\leq$) in each row and in each column. The set of T' satisfying $\lfloor\lambda, T'\rfloor = \lfloor\lambda, T\rfloor$ is exactly $Y(T; W_\lambda)$.*

Let $T_0 \in Y(\tau)$ such that T_0 satisfies condition $[a_i, b_i]_{\mathrm{cm}}$ for each factor $\mathcal{S}_{[a_i,b_i]}$ of W_λ and $a_i \leq j_1 < j_2 \leq b_i$ implies $\mathrm{cm}(j_1, T_0) > \mathrm{cm}(j_2, T_0)$. This condition is equivalent to the tableau $\lfloor\lambda, T_0\rfloor$ being column-strict (the entries strictly increase in each column, see [6, p.5], such tableaux are also called semistandard Young tableaux) and T_0 has a certain extremal property among all $T \in Y(T_0; W_\lambda)$. Let

$$f^s_{\lambda,T_0} = \sum_{\alpha^+=\lambda} \sum_{T\in Y(T_0;W_\lambda)} \prod_{j=1}^{n} P_0\left(T;a_j,b_j\right)\mathcal{E}_-\left(\alpha,T\right)\zeta_{\alpha,T},$$

$$u_{\lambda,T_0} = \sum_{T\in Y(T_0;W_\lambda)} \prod_{j=1}^{n} P_0\left(T;a_j,b_j\right) v_T \in V_\tau.$$

The term involving x^λ is $h_0 = \sum\limits_{T\in Y(T_0;W_\lambda)} \prod\limits_{j=1}^{n} P_0\left(T;a_j,b_j\right)\zeta_{\lambda,T}$, thus the leading term in f^s_{λ,T_0} is $x^\lambda u_{\lambda,T_0}$. From the transformation rules in Proposition 9 it follows that $\|h_0\|^2 = \|\zeta_{\lambda,T_0}\|^2 \|u_{\lambda,T_0}\|_0^2 / \|v_{T_0}\|_0^2$ (see Corollary 2). Also h_0 is W_λ-invariant. In the symbol f^a_{λ,T_0} one could replace T_0 by any $T\in Y\left(T_0;W_\lambda\right)$; then T_0 is the unique solution of $\prod_{j=1}^{n} P_0\left(T;a_j,b_j\right) = 1$.

Theorem 5. *$wf^s_{\lambda,T_0} = f^s_{\lambda,T_0}$ for all $w\in\mathcal{S}_N$ and*

$$\left\|f^s_{\lambda,T_0}\right\|^2 = \frac{N!}{\#W_\lambda}\frac{\|u_{\lambda,T_0}\|_0^2}{\mathcal{E}_+\left(\lambda^R,T_0\right)\|v_{T_0}\|_0^2}\|\zeta_{\lambda,T_0}\|^2.$$

Proof. Let $\mathcal{F}\left(\alpha,T\right) = \prod_{j=1}^{n} P_0\left(T;a_j,b_j\right)\mathcal{E}_-\left(\alpha,T\right)$. Fix $i\in[1,N-1]$ and collect the terms of f^s_{λ,T_0} into three parts. Let

$$\begin{aligned}
L &= \left\{(\alpha,T) : \alpha^+ = \lambda, T\in Y\left(T_0;W_\lambda\right)\right\}\\
A &= \left\{(\alpha,T)\in L : \alpha_i = \alpha_{i+1}, \operatorname{rw}\left(r\left(\alpha,i\right),T\right) = \operatorname{rw}\left(r\left(\alpha,i\right)+1,T\right)\right\},\\
B &= \left\{(\alpha,T)\in L : \alpha_i > \alpha_{i+1}\right\},\\
C &= \left\{(\alpha,T)\in L : \alpha_i = \alpha_{i+1}, \operatorname{rw}\left(r\left(\alpha,i\right),T\right) < \operatorname{rw}\left(r\left(\alpha,i\right)+1,T\right)\right\}.
\end{aligned}$$

The first part is $\sum_{(\alpha,T)\in A}\mathcal{F}\left(\alpha,T\right)\zeta_{\alpha,T}$ and in this case $s_i\zeta_{\alpha,T} = \zeta_{\alpha,T}$. The second part is

$$\begin{aligned}
&\sum_{(\alpha,T)\in B}\left(\mathcal{F}\left(\alpha,T\right)\zeta_{\alpha,T} + \mathcal{F}\left(s_i\alpha,T\right)\zeta_{s_i\alpha,T}\right)\\
&= \sum_{(\alpha,T)\in B}\mathcal{F}\left(\alpha,T\right)\left(\zeta_{\alpha,T} + \frac{\mathcal{F}\left(s_i\alpha,T\right)}{\mathcal{F}\left(\alpha,T\right)}\zeta_{s_i\alpha,T}\right).
\end{aligned}$$

Just as in Proposition 1 $\frac{\mathcal{F}(s_i\alpha,T)}{\mathcal{F}(\alpha,T)} = 1 - b_i\left(\alpha,T\right)$, and hence this sum is s_i-invariant. For use in C let $I\left(\alpha\right) =\operatorname{rw}(\alpha,i)$. Then the third part is

$$\begin{aligned}
&\sum_{(\alpha,T)\in C}\left(\mathcal{F}\left(\alpha,T\right)\zeta_{\alpha,T} + \mathcal{F}\left(\alpha,s_{I(\alpha)}T\right)\zeta_{\alpha,s_{I(\alpha)}T}\right)\\
&= \sum_{(\alpha,T)\in C}\mathcal{F}\left(\alpha,s_{I(\alpha)}T\right)\left(\frac{\mathcal{F}\left(\alpha,T\right)}{\mathcal{F}\left(\alpha,s_{I(\alpha)}T\right)}\zeta_{\alpha,T} + \zeta_{\alpha,s_{I(\alpha)}T}\right).
\end{aligned}$$

To show that each term is s_i-invariant we must show $\frac{\mathcal{F}(\alpha,T)}{\mathcal{F}\left(\alpha,s_{I(\alpha)}T\right)} = b_{I(\alpha)} + 1$. Fix such a term. The equality $\alpha_i = \alpha_{i+1}$ implies $\left[I\left(\alpha\right), I\left(\alpha\right)+1\right]\subset\left[a_i,b_i\right]$ for some i. Thus

$$\frac{\prod_{j=1}^{n} P_0\left(T;a_j,b_j\right)}{\prod_{j=1}^{n} P_0\left(s_{I(\alpha)}T;a_j,b_j\right)} = \frac{P_0\left(T;a_i,b_i\right)}{P_0\left(s_{I(\alpha)}T;a_i,b_i\right)} = 1 + b_{I(\alpha)}\left(T\right).$$

Finally consider $\mathcal{E}_-(\alpha,T)/\mathcal{E}_-\left(\alpha,s_{I(\alpha)}T\right)$; let $g_{lj}(T)=1-\frac{\kappa}{\alpha_j-\alpha_l+\kappa(c(r(\alpha,j),T)-c(r(\alpha,l),T))}$ if $l<j$ and $\alpha_l<\alpha_j$, and $g_{lj}(T)=1$ otherwise. Then $c(r(\alpha,j),T)=c\left(r(\alpha,j),s_{I(\alpha)}T\right)$ whenever $r(\alpha,j)\notin\{I(\alpha),I(\alpha)+1\}$, also $c(r(\alpha,i),T)=c\left(r(\alpha,i+1),s_{I(\alpha)}T\right)$ and $c(r(\alpha,i+1),T)=c\left(r(\alpha,i),s_{I(\alpha)}T\right)$. Thus $g_{l,i}(T)=g_{l,i+1}\left(s_{I(\alpha)}T\right)$ and $g_{l,i}\left(s_{I(\alpha)}T\right)=g_{l,i+1}(T)$ for $1\leq l<i$ with similar relations for g_{ij} and $g_{i+1,j}$ when $i+1<j\leq N$. Also $g_{i,i+1}(T)=1=g_{i,i+1}\left(s_{I(\alpha)}T\right)$ thus $\mathcal{E}_-(\alpha,T)=\prod_{1\leq l<j\leq N}g_{lj}(T)=\mathcal{E}_-\left(\alpha,s_{I(\alpha)}T\right)$. Hence $s_if^s_{\lambda,T_0}=f^s_{\lambda,T_0}$.

To compute $\left\|f^s_{\lambda,T_0}\right\|^2$ consider $\sum\limits_{w\in\mathcal{S}_N}wh_R$ where $h_R=\sum\limits_{T\in Y(T_0;W_\lambda)}\prod\limits_{j=1}^{n}P_0(T;a_j,b_j)\,\zeta_{\lambda^R,T}$. By the argument used above for type (4) $\mathcal{E}_-\left(\lambda^R,T\right)=\mathcal{E}_-\left(\lambda^R,T_0\right)$ for all $T\in Y(T_0;W_\lambda)$. Thus the term for $\alpha=\lambda^R$ in f^s_{λ,T_0} is $\mathcal{E}_-\left(\lambda^R,T_0\right)h_R$ and leading term in h_R is $x^{\lambda^R}w_{\lambda^R}u_{\lambda,T_0}$. Similarly to the proof of Theorem 3 we conclude $\sum\limits_{w\in\mathcal{S}_N}wh_R=(\#W_\lambda)\,f^s_{\lambda,T_0}$ and $(\#W_\lambda)\left\|f^s_{\lambda,T_0}\right\|^2=N!\left\langle h_R,f^s_{\lambda,T_0}\right\rangle=N!\mathcal{E}_-\left(\lambda^R,T_0\right)\|g\|^2$. Finally $\|g\|^2=\left\|\zeta_{\lambda^R,T_0}\right\|^2\|u_{\lambda,T_0}\|_0^2/\|v_{T_0}\|_0^2=\dfrac{\|\zeta_{\lambda,T_0}\|^2\|u_{\lambda,T_0}\|_0^2}{\mathcal{E}_2(\lambda^R,T_0)\|v_{T_0}\|_0^2}$. □

Corollary 5. *Suppose $\lambda,\mu\in\mathbb{N}_0^{N,+}$ and $T_1,T_2\in Y(\tau)$ such that $\lfloor\lambda,T_1\rfloor$ and $\lfloor\mu,T_2\rfloor$ are column-strict. If $\lambda\neq\mu$ or $T_2\notin Y(T_1;W_\lambda)$ then $\left\langle f^s_{\lambda,T_1},f^s_{\mu,T_2}\right\rangle=0$.*

Let $T_0\in Y(\tau)$ such that T_0 satisfies condition $[a_i,b_i]_{\text{rw}}$ for each factor $\mathcal{S}_{[a_i,b_i]}$ of W_λ and $a_i\leq j_1<j_2\leq b_i$ implies $\text{cm}(j_1,T_0)\leq\text{cm}(j_2,T_0)$. This condition is equivalent to the tableau $\lfloor\lambda,T\rfloor$ being row-strict (the entries strictly increase in each row), and T_0 having a certain extremal property. Let

$$f^a_{\lambda,T_0}=\sum_{\alpha^+=\lambda}(-1)^{\text{inv}(\alpha)}\sum_{T\in Y(T_0;W_\lambda)}\prod_{j=1}^{n}P_1(T;a_j,b_j)\,\mathcal{E}_+(\alpha,T)\,\zeta_{\alpha,T},$$

$$u_{\lambda,T_0}=\sum_{T\in Y(T_0;W_\lambda)}\prod_{j=1}^{n}P_1(T;a_j,b_j)\,v_T\in V_\tau.$$

The term involving x^λ is $h_0=\sum\limits_{T\in Y(T_0;W_\lambda)}\prod\limits_{j=1}^{n}P_1(T;a_j,b_j)\,\zeta_{\lambda,T}$, thus the leading term in f^a_{λ,T_0} is $x^\lambda u_{\lambda,T_0}$. From the transformation rules in Proposition 9 it follows that $\|h_0\|^2=\|\zeta_{\lambda,T_0}\|^2\|u_{\lambda,T_0}\|^2/\|v_{T_0}\|^2$ (see Proposition 4). Also h_0 is W_λ-antisymmetric.

Theorem 6. *$wf^a_{\lambda,T_0}=sgn(w)\,f^a_{\lambda,T_0}$ for all $w\in\mathcal{S}_N$ and*

$$\left\|f^a_{\lambda,T_0}\right\|^2=\frac{N!}{\#W_\lambda}\frac{\|u_{\lambda,T_0}\|_0^2}{\mathcal{E}_-(\lambda^R,T_0)\|v_{T_0}\|_0^2}\|\zeta_{\lambda,T_0}\|^2.$$

Proof. Let $\mathcal{F}(\alpha,T)=\prod_{j=1}^{n}P_1(T;a_j,b_j)\,\mathcal{E}_+(\alpha,T)$. Fix $i\in[1,N-1]$ and collect the terms of f_{λ,T_0}^{a} into three parts. Let

$$\begin{aligned}
L&=\left\{(\alpha,T):\alpha^+=\lambda,T\in Y(T_0;W_\lambda)\right\}\\
A&=\{(\alpha,T)\in L:\alpha_i=\alpha_{i+1},\operatorname{cm}(r(\alpha,i),T)=\operatorname{cm}(r(\alpha,i)+1,T)\},\\
B&=\{(\alpha,T)\in L:\alpha_i>\alpha_{i+1}\},\\
C&=\{(\alpha,T)\in L:\alpha_i=\alpha_{i+1},\operatorname{rw}(r(\alpha,i),T)<\operatorname{rw}(r(\alpha,i)+1,T)\}.
\end{aligned}$$

□

The proof that each of the following satisfies $s_if=-f$ is analogous to the proof of the previous theorem:

$$\begin{aligned}
&\sum_{(\alpha,T)\in A}(-1)^{\operatorname{inv}(\alpha)}\mathcal{F}(\alpha,T)\,\zeta_{\alpha,T},\\
&\sum_{(\alpha,T)\in B}(-1)^{\operatorname{inv}(\alpha)}\left(\mathcal{F}(\alpha,T)\,\zeta_{\alpha,T}-\mathcal{F}(s_i\alpha,T)\,\zeta_{s_i\alpha,T}\right)\\
&=\sum_{(\alpha,T)\in B}(-1)^{\operatorname{inv}(\alpha)}\mathcal{F}(\alpha,T)\left(\zeta_{\alpha,T}-\frac{\mathcal{F}(s_i\alpha,T)}{\mathcal{F}(\alpha,T)}\zeta_{s_i\alpha,T}\right),\\
&\sum_{(\alpha,T)\in C}(-1)^{\operatorname{inv}(\alpha)}\left(\mathcal{F}(\alpha,T)\,\zeta_{\alpha,T}+\mathcal{F}\left(\alpha,s_{I(\alpha)}T\right)\zeta_{\alpha,s_{I(\alpha)}T}\right)\\
&=\sum_{(\alpha,T)\in C}(-1)^{\operatorname{inv}(\alpha)}\mathcal{F}\left(\alpha,s_{I(\alpha)}T\right)\left(\frac{\mathcal{F}(\alpha,T)}{\mathcal{F}\left(\alpha,s_{I(\alpha)}T\right)}\zeta_{\alpha,T}+\zeta_{\alpha,s_{I(\alpha)}T}\right).
\end{aligned}$$

In the second equation $\frac{\mathcal{F}(s_i\alpha,T)}{\mathcal{F}(\alpha,T)}=1+b_i(\alpha,T)$. In the third equation $I=r(\alpha,i)$ and $\frac{\mathcal{F}(\alpha,T)}{\mathcal{F}\left(\alpha,s_{I(\alpha)}T\right)}=b_{I(\alpha)}-1$. The proof for the norm formula is also analogous, based on $\sum_{w\in\mathcal{S}_N}wh_R$ where $h_R=\sum\limits_{T\in Y(T_0;W_\lambda)}\prod\limits_{j=1}^{n}P_1(T;a_j,b_j)\,\zeta_{\lambda^R,T}$. Note $\operatorname{sgn}(w_{\lambda^R})=(-1)^{\operatorname{inv}\left(\lambda^R\right)}$.

Remark 2. *The polynomials in Theorems 5 and 6 form orthogonal bases for the symmetric and antisymmetric polynomials, respectively, in $M(\tau)$.*

We now establish the striking results concerning the norms of certain symmetric and antisymmetric polynomials. For a given partition τ of N there are unique symmetric and antisymmetric polynomials of minimum degree in the standard module $\mathcal{M}(\tau)$. It is obvious that the column-strict tableau $\langle\lambda,T\rangle$ with minimum $|\lambda|$ has the entries 0 in row #1, 1 in row #2 and so on (consider the minimum entries in each column). Denote this partition by $\delta^s(\tau)$ and the unique possible T by T^s (the entries $N,N-1,\ldots,2,1$ are entered row-by-row in the Ferrers diagram of τ). Example: let $\tau=(5,3,2)$ then

$$T^s=\begin{array}{ccccc}10&9&8&7&6\\5&4&3&&\\2&1&&&\end{array},\ \lfloor\delta^s(\tau),T^s\rfloor=\begin{array}{ccccc}0&0&0&0&0\\1&1&1&&\\2&2&&&\end{array},$$

and $\delta^s(\tau)=(2,2,1,1,1,0,0,0,0,0)$.

Similarly the row-strict tableau $\langle \lambda, T\rangle$ with minimum $|\lambda|$ has the entries 0 in column #1, 1 in column #2 and so on (consider the minimum entries in each row). Denote this partition by $\delta^a(\tau)$ and the unique possible T by T^a (the entries $N, N-1, \ldots, 2, 1$ are entered column-by-column in the Ferrers diagram of τ). Example: let $\tau = (5,3,2)$ then

$$T^a = \begin{array}{ccccc} 10 & 7 & 4 & 2 & 1 \\ 9 & 6 & 3 & & \\ 8 & 5 & & & \end{array}, \lfloor \delta^a(\tau), T^a \rfloor = \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & & \\ 0 & 1 & & & \end{array},$$

and $\delta^a(\tau) = (4,3,2,2,1,1,1,0,0,0)$. The sum of the hook-lengths of τ equals $|\delta^s(\tau)| + |\delta^a(\tau)| + N$ (see [6, Ex.2, p.11]).

Let $f^s_\tau = f^s_{\delta^s(\tau),T^s}$ and $f^a_\tau = f^a_{\delta^a(\tau),T^a}$. These polynomials are actually independent of κ; there is no composition α such that $\alpha \lhd \delta^s(\tau)$ and $\alpha^+ \neq \delta^s(\tau)$ which can occur in a symmetric polynomial, due to the minimality of $\delta^s(\tau)$. A similar argument applies to $\delta^a(\tau)$. To compute the norms of $\|f^s_\tau\|^2$ and $\|f^a_\tau\|^2$ we use the special properties of $\delta^s(\tau)$ to write simplified formulae. To use the formulae in Theorems 2 and 3 note that $\delta_a(\tau)_j = i-1$ when j appears in row $\#i$ of T^s, and the corresponding contents of T^s are $1-i, \ldots, \tau_i - i$. Let $L = \ell(\tau)$ and

$$\begin{aligned}
P_1(\tau) &= \prod_{i=2}^{L}\prod_{j=1}^{\tau_i} (1+\kappa(j-i))_{i-1}, \\
P_2(\tau) &= \prod_{1\le i<j\le L}\prod_{j_1=1}^{\tau_i}\prod_{j_2=1}^{\tau_j}\prod_{r=1}^{j-i}\left(1-\frac{\kappa^2}{(r+\kappa(j_2-j_1-j+i))^2}\right), \\
P_3(\tau) &= \prod_{1\le i<j\le L}\prod_{j_1=1}^{\tau_i}\prod_{j_2=1}^{\tau_j}\left(1+\frac{\kappa}{j-i+\kappa(j_2-j_1-j+i)}\right); \\
H^s(\tau) &= \prod_{(i,j)\in\tau} (1-\kappa h(i,j))_{\mathrm{leg}(i,j)}.
\end{aligned}$$

Then $\left\|\zeta_{\delta^s(\tau),T^s}\right\|^2 = \|v_{T^s}\|^2 P_1(\tau) P_2(\tau)$ and $\mathcal{E}_+\left(\delta^s(\tau)^R, T^s\right) = P_3(\tau)$.

Theorem 7. *Suppose τ is a partition then* $\dfrac{P_1(\tau)P_2(\tau)}{P_3(\tau)} = H^s(\tau)$.

Proof. We use induction on the last part τ_L. The induction begins with $\tau = (N)$ where each product equals 1. Let $\sigma = (\tau_1, \tau_2, \ldots, \tau_L - 1)$ and assume the formula is valid for σ. (It is possible that $\tau_L = 1$ and $\ell(\sigma) = L-1$). The nodes in σ and τ have the same hook-lengths except for the nodes (i,L) with $1 \le i < L$ and (L,j) with $1 \le j \le \tau_L$. The latter have zero leg-length and do not contribute to $H^s(\sigma)$ or $H^s(\tau)$. Then for $1 \le i < L$

$$\begin{aligned}
\mathrm{arm}(i,L;\sigma) &= \mathrm{arm}(i,L;\tau) = \tau_i - \tau_L \\
\mathrm{leg}(i,L;\sigma)+1 &= \mathrm{leg}(i,L;\tau) = L-i \\
h(i,L;\sigma)+1 &= h(i,L;\tau) = 1+\tau_i-\tau_L+L-i,
\end{aligned}$$

and thus

$$\frac{H^s(\tau)}{H^s(\sigma)} = \prod_{i=1}^{L-1}\frac{(1-\kappa(\tau_i-\tau_L+L-i+1))_{L-i}}{(1-\kappa(\tau_i-\tau_L+L-i))_{L-i-1}}.$$

Firstly,

$$\frac{P_1(\tau)}{P_1(\sigma)} = (1+\kappa(\tau_L - L))_{L-1};$$

secondly

$$\begin{aligned}
\frac{P_2(\tau)}{P_2(\sigma)} &= \prod_{i=1}^{L-1} p_i(\tau),\\
p_i(\tau) &= \prod_{r=1}^{L-i}\prod_{j=1}^{\tau_i} \frac{(r+\kappa(\tau_L - L + i) - (j-1)\kappa)(r+\kappa(\tau_L - L+i) - (j+1)\kappa)}{(r+\kappa(\tau_L - L+i) - j\kappa)(r+\kappa(\tau_L - L+i) - j\kappa)}\\
&= \prod_{r=1}^{L-i} \frac{(r+\kappa(\tau_L - L+i))(r+\kappa(\tau_L - L+i) - (\tau_i+1)\kappa)}{(r+\kappa(\tau_L - L+i) - \kappa)(r+\kappa(\tau_L - L+i) - \tau_i\kappa)}\\
&= \frac{(1+\kappa(\tau_L - L+i))_{L-i}(1+\kappa(\tau_L - \tau_i - L+i-1))_{L-i}}{(1+\kappa(\tau_L - L+i-1))_{L-i}(1+\kappa(\tau_L - \tau_i - L+i))_{L-i}};
\end{aligned}$$

a telescoping product argument is used to produce the third line from the second. Thirdly,

$$\begin{aligned}
\frac{P_3(\sigma)}{P_3(\tau)} &= \prod_{i=1}^{L-1}\prod_{j_1=1}^{\tau_i}\left(\frac{L-i+\kappa(\tau_L - j_1 - L+i)}{L-i+\kappa(\tau_L+1-j_1-L+i)}\right)\\
&= \prod_{i=1}^{L-1}\frac{L-i+\kappa(\tau_L - \tau_i - L+i)}{L-i+\kappa(\tau_L - L+i)}.
\end{aligned}$$

Combining these products and by use of

$$\begin{aligned}
\frac{L-i+\kappa(\tau_L - \tau_i - L+i)}{(1+\kappa(\tau_L - \tau_i - L+i))_{L-i}} &= \frac{1}{(1+\kappa(\tau_L - \tau_i - L+i))_{L-i}},\\
\frac{(1+\kappa(\tau_L - L+i))_{L-i}}{L-i+\kappa(\tau_L - L+i)} &= (1+\kappa(\tau_L - L+i))_{L-i-1},
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{P_1(\tau)P_2(\tau)P_3(\sigma)H^s(\sigma)}{P_1(\sigma)P_2(\sigma)P_3(\tau)H^s(\tau)} &= (1+\kappa(\tau_L - L))_{L-1}\prod_{i=1}^{L-1}\frac{(1+\kappa(\tau_L - L+i))_{L-i-1}}{(1+\kappa(\tau_L - L+i-1))_{L-i}}\\
&= 1.
\end{aligned}$$

The last step is actually easy: replace i by $i-1$ in the numerator (and now $2 \leq i \leq L$) and cancel. This completes the induction. □

Theorem 8. *Suppose τ is a partition of N then*

$$\|f_\tau^s\|^2 = \frac{N!}{\prod_{i=1}^{\tau_1}\tau_i'!}\|v_{T^s}\|_0^2 \prod_{(i,j)\in\tau}(1-\kappa h(i,j))_{\mathrm{leg}(i,j)}.$$

Proof. The formulae of Theorem 5 and the previous Theorem imply this result. The stabilizer subgroup $W_{\delta^s(\tau)}$ is acts on the columns of τ and has order $\tau_1'!\tau_2'!\ldots$. □

Theorem 9. *Suppose τ is a partition of N then*

$$\|f_\tau^a\|^2 = \frac{N!}{\prod_{i=1}^{\ell(\tau)}\tau_i!}\|v_{T^a}\|_0^2 \prod_{(i,j)\in\tau}(1+\kappa h(i,j))_{\mathrm{arm}(i,j)}.$$

Proof. Apply Theorem 6 and the formula in Theorem 8 to the conjugate τ' of τ and with κ replaced by $-\kappa$. Then $\operatorname{leg}(j,i;\tau') = \operatorname{arm}(i,j;\tau)$ for $(i,j) \in \tau$. Note however that $\|v_{T^s}\|_0^2 / \|v_{T^a}\|_0^2$ is computed by use of Proposition 4. $\square$

As example we use $\tau = (5,3,2)$ again. The hook-lengths and norms are

$$\begin{array}{ccccc} 7 & 6 & 4 & 2 & 1 \\ 4 & 3 & 1 & & \\ 2 & 1 & & & \end{array},$$

$$\|f_\tau^s\|^2 = c_0 (1-7\kappa)_2 (1-6\kappa)_2 (1-4\kappa)^2 (1-3\kappa),$$
$$\|f_\tau^a\|^2 = c_1 (1+7\kappa)_4 (1+6\kappa)_3 (1+4\kappa)_2^2 (1+3\kappa)(1+2\kappa)^2.$$

Analogously to the $M_{(N)}$ (trivial representation) result, each hook-length m appears in $m-1$ factors $(m\kappa + r)$ involving each nonzero residue class $\operatorname{mod} m$. In the example for $m = 6$ we obtain $6\kappa-2, 6\kappa-1, 6\kappa+1, 6\kappa+2, 6\kappa+3$. We conjecture that the singular values for $M(\tau)$ form a subset of $\left\{\frac{n}{m} : m = h(i,j), (i,j) \in \tau, \frac{n}{m} \notin \mathbb{Z}\right\}$ ($\kappa_0 \in \mathbb{Q}$ is a singular value if there exists nonzero $f \in M(\tau)$ such that $\mathcal{D}_i(\kappa_0) f = 0$ for all $i \in [1,N]$; that is, the generic κ is specialized to κ_0; the condition is equivalent to $J_{\kappa_0}(\tau) \neq (0)$). As yet there is insufficient evidence for speculation about any further restrictions.

S. Griffeth (personal communication, gratefully acknowledged) points out that Theorem 8 provides a new proof for one of the parts of the Gordon-Stafford Theorem [4, Cor. 3.13]; another proof was found by Bezrukavnikov and Etingof [1, Cor. 4.2]; note that these papers use $c = -\kappa$ as parameter. An *aspherical* module of the rational Cherednik algebra is one containing no nonzero $\mathcal{S}_N$-invariant. If some quotient module of a standard module is aspherical for a numerical value κ_0 of κ then κ_0 is called an *aspherical value*. Theorem 8 shows that any aspherical value is in $\left\{\frac{m}{n} : 1 \leq m < n \leq N\right\}$ (this is one component of the Gordon-Stafford theorem, which deals with the problem of Morita equivalence of rational Cherednik algebras for parameters κ and $\kappa - 1$). Suppose M_0 is a proper submodule of $M(\tau)$ for $\kappa = \kappa_0 \in \mathbb{Q}$ (that is, a specific numerical value). This means that M_0 is closed under multiplication by x_i and the action of $\mathcal{D}_i$ for $i \in [1,N]$ and under the action of $\mathcal{S}_N$. Then $f \in M_0$ implies $\langle g, f\rangle = 0$ for all $g \in M(\tau)$ (M_0 is a submodule of the radical $J_{\kappa_0}(\tau)$, the maximal submodule.). Indeed, by the definition of the contravariant form, $\langle x^\alpha u, f\rangle = \langle u, \mathcal{D}^\alpha f(x)|_{x=0}\rangle_0$ for $\alpha \in \mathbb{N}_0^N, u \in V_\tau$ (and $\mathcal{D}^\alpha = \prod_{i=1}^N \mathcal{D}_i^{\alpha_i}$). If $f \in \mathcal{P}_n \otimes V_\tau$ and $|\alpha| = n$ then $\mathcal{D}^\alpha f(x) \in V_\tau$. If also $f \in M_0$ then $\mathcal{D}^\alpha f(x) = 0$, or else $M_0 = M(\tau)$. If $M(\tau)/M_0$ is aspherical then $f_\tau^s \in M_0$ and $\kappa = \kappa_0$ is a zero of $\prod_{(i,j)\in\tau} (1 - \kappa h(i,j))_{\operatorname{leg}(i,j)}$.

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